Week 9 :, Chapter - 5
Eigen values and Eigen vectors

Defn: Let A be a square matrix of order n, then the value of λ for which the equation $Ax = \lambda x$ —(1) has a non-trivial solution are called Eigen value of A.

corresponding Eigen value λ there exist a non-zero vector X. such that $|A - \lambda I| X = 0$ Then X is called The Eigen vector.

Remark (1) $|A - \lambda I| = 0$ is called the characteristic equation (in some book) $|\lambda I - A| = 0$ or $\det(\lambda I - A)$

(2) The Eigen value of triangular matrix are its diagonal elements.

(3) A is singular matrix $|A| = 0 \Leftrightarrow \lambda = 0$

Q1. Determine the Eigen value and the corresponding Eigen vector of $A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$

Soln: **Step 1:** To find the Eigen value first write the characteristic equation

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -1 \\ -6 & \lambda - 2 \end{vmatrix} = 0 \qquad \text{—(1)}$$

$$(\lambda - 3)(\lambda - 2) - 6 = 0$$
$$\lambda^2 - 2\lambda - 3\lambda + 6 - 6 = 0$$
$$\lambda^2 - 5\lambda = 0$$
$$\lambda(\lambda - 5) = 0$$
$$\lambda = 0, 5$$

Thus $\lambda_1 = 0$ and $\lambda_2 = 5$ are the Eigen values of A.

step 2: To find the Eigen vector

(i) Eigen vector corresponding to $\lambda_1 = 0$

$(\lambda_1 I - A) X = 0$ Put $\lambda_1 = 0$ in (1)

$$\Rightarrow \begin{pmatrix} 0-3 & -1 \\ -6 & 0-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -3 & -1 \\ -6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad R_2 \to R_2 + 2R_1$$

$$\begin{pmatrix} -3 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Since the The rank of the matrix is one. So there is only one Leading variable ie x_1 and one free variable ie x_2. So we can write

$$-3x_1 - 1x_2 = 0 \quad \text{or}$$

$$-3x_1 = x_2$$

Let us assume $x_2 = 3$ Then $x_1 = -\frac{3}{3} = -1$

So Eigen vector is $(-1, 3)$

Let us assume $x_2 = 6$ Then $x_1 = -\frac{6}{3} = 2$

So another Eigen vector may be $(-2, 6)$ and so on.

(ii) Eigen vector corresponding to $\lambda = 5$

we have $(\lambda I - A) X = 0$ Put in (1)

$$\begin{bmatrix} 5-3 & -1 \\ -6 & 5-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$R_2 \to R_2 + 3R_1$
Since Rank is one
$2x_1 - x_2 = 0$
Leading variable x_1
free variable x_2

$2x_1 = x_2$ Let $x_2 = 2$ Then $x_1 = \frac{2}{2} = 1$ $(1, 2)$ is the Eigen vector

Similarly Let $x_2 = 4$ Then ...

Q2. Find the Eigen value of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

Soln: The characteristic equation is

$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda - (-1) \end{vmatrix} = 0$$

$$(\lambda - 3)\quad(\lambda + 1) = 0$$

So $\lambda = 3$ and $\lambda = -1$ are the Eigen value of A.

Q3. Find the Eigen value of $A = \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 2/3 & 0 \\ 5 & -8 & -1/4 \end{bmatrix}$

Soln: Since it is a Lower triangular matrix. So by the property its diagonal elements are the Eigen value:

$$\lambda_1 = 1/2 \quad , \lambda_2 = 2/3 \qquad \lambda_3 = 1/4$$

Eigen value of 3×3 Matrix

Q4. Find the Eigen value of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

Soln: The characteristic equation is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 0 & -1 & -0 \\ -0 & \lambda - 0 & -1 \\ -4 & 17 & \lambda - 8 \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{vmatrix} = 0$$

open the determinant along Row side

$$= \lambda \begin{vmatrix} \lambda & -1 \\ 17 & \lambda - 8 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ -4 & \lambda - 8 \end{vmatrix} + 0 \left|\quad\right| = 0$$

$$\lambda\left[(\lambda^2 - 8\lambda) + 17\right] + (0 - 4) + 0 = 0$$

$$\Rightarrow \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0 \quad -(1)$$

Now check $\lambda = \pm 1, \pm 2, \pm 3, \pm 4$ successively substitute these value in (1) so we observe at $\lambda = 4$ it will satisfy (1) so $(\lambda - 4)$ is one solution Now

$$\begin{array}{r|l} & \lambda^2 - 4\lambda + 1 \\ \hline \lambda - 4 & \lambda^3 - 8\lambda^2 + 17\lambda - \ldots \\ & \lambda^3 - 4\lambda^2 \\ & -\ \ + \\ \hline & -4\lambda^2 + 17\lambda \\ & -4\lambda^2 + 16\lambda \\ & +\ \ - \\ \hline & \lambda - \ldots \\ & \lambda - 4 \\ & -\ \ + \\ \hline & 0\ 0 \end{array}$$

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

Now I am going to factorise with $\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\lambda = 4, \quad \lambda = 2 + \sqrt{3}, \quad \lambda = 2 - \sqrt{3}$$

are the Eigen value.

(3)

Q Find the Eigen value, Eigen vector and diagonalize the matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Soln: The characteristic equation is $|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} = 0 \quad \text{—(1)}$$

on solving $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

check for $\lambda = 1$: $\lambda^3 - 5(1)^2 + 8(1) - 4 = 0$

$0 = 0$

$(\lambda - 1)$ is a factor. Now

$$\lambda - 1 \overline{)\ \lambda^3 - 5\lambda^2 + 8\lambda - 4\ } (\lambda^2 - 4\lambda + 4$$

$\underline{\pm\lambda^3 \mp \lambda^2}$

$-4\lambda^2 + 8\lambda$

$\underline{-4\lambda^2 + 4\lambda}$

$4\lambda - 4$

$\underline{4\lambda \mp 4}$

0

So

$(\lambda-1)(\lambda^2 - 4\lambda + 4)$

$(\lambda-1)(\lambda-2)(\lambda-2)$

$(\lambda-1)(\lambda-2)^2$

$\lambda_1 = 1,\ \lambda_2 = 2, 2$

are the Eigen value

Eigen vector: corresponding to $\lambda = 1$

$(\lambda I - A)X = 0$ from (1)

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad R_2 \to R_2 + R_1,\ R_3 \to R_3 + R_1$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Since the rank of the matrix is 2. There are two leading variable and one free variable: Leading x_1 and x_2, free x_3

Now we can write

$x_1 + 0x_2 + 2x_3 = 0$ or $x_1 + 2x_3 = 0$ —(2)

$-x_2 + x_3 = 0$ $\qquad x_3 = x_2$ —(3)

Let $x_3 = 1$ in (3) then $x_2 = 1$ and from (2) $x_1 = -2x_3 \Rightarrow x_1 = -2$