Week 7, chapter 4 (4.7 – 4.11)

Row Space, Column Space and Null space

Row and colum vectors of a 2×3 Matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

The Row vectors of A are

$r_1 = [2 \quad 1 \quad 0]$ — 3 vectors and $r_2 = [3 \quad -1 \quad 4]$ — 3 vectors

and The column vectors of A are

$c_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ — 2 vectors $c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $c_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

Now we are going to define 3 important vector spaces that are associated with a matrix

Definition: If A is an $m \times n$ matrix, Then the subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the row space of A, and the subspace of $\mathbb{R}^m$ spamed by the column vectors of A is called The column space of A.

The solution space of the homogeneous system of eqns $Ax = 0$ which is a subspace of $\mathbb{R}^n$, is called the null space of A.

Q1 Find the basis for Row and colum spaces of the matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Soln: Since the matrix already is in Row echelon form. Therefore The non-zero vectors are In The first Row the

$r_1 = (1 \quad -2 \quad 5 \quad 0 \quad 3)$

$r_2 = (0 \quad 1 \quad 3 \quad 0 \quad 0)$

$r_3 = (0 \quad 0 \quad 0 \quad 1 \quad 0)$

Now find the basis for the column space.

In each row the first Non-zero element is called the leading element and it is 1, and corresponding to this leading element The variable is called leading variable, and rest of the variable are called the free variables.

So

$$c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad c_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

are called the basis for the column space.

Q2. Given the 3×4 matrix, find The basis for Row space of A. and its dimension

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

Soln: convert into Row echelon form

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

So the basis is $r_1' = [1, 4, 5, 2]$ $\quad r_2' = (0, 1, 1, \frac{4}{7})$
and the dimension for the raw space of A is 2.

$r_1 = [1 \;\; 4 \;\; 5 \;\; 2]$ $\quad r_2 = [2 \;\; 1 \;\; 3 \;\; 0]$

Q3. Determine the dimension of A. and the Basis for Row and Column space of A.

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

Soln: Conver into Row echelon form

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The Rank of A is 3 (No. of non-Rows)
so $\dim(A) = 3$

Now Basis for Row space $r_1' = [1 \;\; 0 \;\; 1]$, $r_2' = [0 \;\; 1 \;\; -1$

And Another Basis for Row space are original rows corresponding to these rows

$r_1 = [2\ -1\ 3]$, $r_2 = [1\ 0\ 1]$ $r_3 = [0\ 2\ -1]$

Similarly

$$c_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad c_2' = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad c_3' = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

are the column spaces. and corresponding to the original matr

$$c_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 4 \end{bmatrix}$$

Qu. Find a basis for the null space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 2 & 6 & -5 & -2 \\ 0 & 0 & 5 & 10 \\ 2 & 6 & 0 & 8 \end{bmatrix}$$

Sol: Reduce the given matrix into row echelon form

$$\begin{pmatrix} 1 & 3 & -2 & 0 \\ 2 & 6 & -5 & -2 \\ 0 & 0 & 5 & 10 \\ 2 & 6 & 0 & 8 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix} \begin{pmatrix} 1 & 3 & -2 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 4 & 8 \end{pmatrix} \begin{matrix} R_3 \to 5R_2 + \\ R_4 \to 5R_2 + \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now for the null space: The solution space of the homogeneous system of eqn $AX = 0$ which is subspace of R^n

$$\begin{pmatrix} 1 & 3 & -2 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

— Leading variable (coz the coefficient of x is 1 & its leading in 1st row)

$x + 3y - 2z = 0$ — (1)

Leading variable

$-z - 2w = 0$ — (2)

∴ Here free variables are y and w

∴ dimension of null space is 2.

Since y & w are free so we can take any value of

(1) Let Put $y = 1$ and $w = 0$:

$\cancel{x + 3 \times 1 - 2}$

from (2) $-z - 2 \times 0 = 0$

$\Rightarrow z = 0$

Put in (1)

$x + 3 \times 1 - 2 \times 0 = 0$

$x + 3 = 0$

$x = -3$

∴ $(-3, 1, 0, 0)$ is the soln.

(2) Let $y = 0$ and $w = 1$:

from (2) $-z - 2 \times 1 = 0$

$-z = 2$

$z = -2$

and $x + 0 + 4 = 0$

$x = -4$

∴ $(-4, 0, -2, 0)$ is the soln.

Therefore $\{(-3, 1, 0, 0), (-4, 0, -2, 0)\}$ is the basis for null space.

Rank and Nullity:

Remark: The row space and column space of a matrix A have the same dimension.

Defn: The common dimension of the row and column space of a matrix A is called the rank of A and it is denoted by rank (A). The dimension of the null space of A is called the nullity of A and is denoted by nullity (A)

Dimension Theorem for Matrix

If A is a matrix with n columns, then

rank (A) + nullity (A) = n

Q. Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Soln: Reduce this matrix to Row echelon form

$$\begin{pmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & 1 & 2 & 12 & 16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank (A) = 2 (No. of non zero Rows)

Now By R. Nullity Theorem

rank (A) + nullity (A) = No. of columns in the matrix

2 + nullity (A) = 6

nullity (A) = 6 − 2 = 4

Q: Find the rank and Nullity of the matrix A.

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

Soln: convert into Row echelon form

$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & -12 \\ 0 & -7 & -1 & 12 \end{bmatrix} \quad R_2 \to \frac{R_2}{7}$$

$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1/7 & \frac{-12}{7} \\ 0 & 7 & -1 & 12 \end{bmatrix} \quad R_3 \to R_3 + 7R_2$$

$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1/7 & -\frac{12}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 + 2R_1 \quad \text{(Row echelon form)}$$

$$\begin{bmatrix} 1 & 0 & \frac{2}{7} & \frac{4}{7} \\ 0 & 1 & 1/7 & -\frac{12}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Reduced Row echelon form}$$

Since Rank(A) = 2 (No. of non zero Rows)

$n = 4$ (No. of columns)

By Rank Nullity Theorem

$$\text{rank}(A) + \text{nullity}(A) = n$$

$$2 + \text{nullity}(A) = 4$$

$$\text{nullity}(A) = 4 - 2$$

$$= 2$$

Q3 Let $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix}$ (4)

(a) Find the rank and nullity of A
(b) Find a subset of the column vectors of that forms a basis for the column space of A.

Sol: (a) convert into Row echelon form

first non-zero Leading 1

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3

By Rank nullity Theorem

$$R(A) + \text{nullity}(A) = n$$

$$3 + \text{nullity}(A) = 5$$

$$\text{nullity}(A) = 5 - 3 = 2$$

(b) The basis for column space are " The Pivotal columns"

$$c_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_2' = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad c_4' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Corresponding to these columns

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \quad c_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$