Week 6, chapter 4

General Vector Spaces

n - space : $R^n \rightarrow$ The set of all ordered n-tuple (a sequence of n real number $(x_1, x_2, \dots x_n)$)

for ex: $n = 1 \rightarrow R^1$ = 1 - space
= set of all real numbers

$n = 2 \rightarrow R^2$ = 2 - space
= set of all ordered pair of real numbers (x_1, x_2)

$n = 3 \rightarrow R^3$ = 3 space
= set of all ordered triple of real numbers (x_1, x_2, x_3)

⋮

Vector space: Let V be a non-empty set on which two operations addition and scalar multiplication are defined $(V, +, \cdot)$ is said to vector space if all the following axioms are satisfied.

for all $u, v, w \in V$ and all scalars k and m

Addition:

(1) $u + v \in V$

(2) $u + v = v + u$

(3) $u + (v + w) = (u + v) + w$

(4) $0 + u = u + 0 = u$

(5) $u + (-u) = (-u) + u = 0$

Scalar Multiplication

(6) If k is any scalar and $u \in V$, then $ku \in V$

(7) $k(u + v) = ku + kv$

(8) $(k + m) u = ku + mu$

(9) $k(mu) = (km) u$

(10) $1 \cdot u = u$

Note: (1) A vector space consist of 4 entities
(1) a set of vector (2) a set of scaler
(3) two operations (addition & Multip)
or
(1) V: non empty
(2) c: scalar
(3) vector addition
(4) scalar multiplication

$(V, +, \cdot)$ is a vector space

(2) $V = \{0\}$ zero vector space

(3) To show that a set is not a vector space, you need only find one axiom that is not satisfied.

(4) How to check the V-S
step 1: Identify the set V of objects that will become vectors
step 2: Verify Axiom 1 and Axiom 6 ie addition & scalar multiplication.
step 3: confirm that ~~Axioms~~ rest of the axioms 2, 3, 4, 5, 7, 8, 9, 10 hold.

Q1 Zero is a vector space
(1) Define addition & scalar multiplication
$0 + 0 = 0$
and $k \cdot 0 = 0$
$\therefore$ 0 is a vector space. we can check all the vector space axioms are satisfied

Q2. The set of all Integers is not a vector space
Soln! Let $1 \in V$ — vector space, $\frac{1}{2} \in R$ — Real
Define the addition & scaler multiplication as
(1) $\underset{\text{Integer}}{1} + \underset{\text{Integer}}{2} = 3$ → Integer. It is closed under vector addition
(2) $\frac{1}{2}(1) = \frac{1}{2} \notin V$ It is not closed under scalar multiplication

Q3. The set of all second degree polynomial is not a vector space ②

Soln: Let $P(x) = x^2$

$$q(x) = -x^2 + x + 1$$

(i) vector addition

$$P(x) + q(x) = \cancel{x^2} + \cancel{(-x^2)} + x + 1$$

$$= x + 1 \notin V$$

it is not closed under vector addition.

Subspace: A subset W is a vector space V is called a subspace of V, if W is itself a vector space under the addition and scalar multiplication defined in V.

Note: (1) Every vector space V has atleast two subspaces

(1) Zero vector space $\{0\}$ is a subspace of V

(2) V is a subspace of V

W V

Note (2) If W is a non empty subset of a v.s. V, Then W is a subspace of V iff the following conditions holds.

(1) If u and v are in W, then u + v is in W

(2) If u in W and c is any scalar, then cu is in W.

Ex: Show that $U = \{(a,b,c) \mid a = b = c\}$ is a subspace in $\mathbb{R}^3$

Soln: clearly $U = \{\phi\}$ as $(0,0,0) \in U$ $(\because 0 = 0 = 0)$

Take any $u, v \in U$ Then

$$U = (a,a,a) \qquad v = (b,b,b)$$

Now

① $$u + v = (a,a,a) + (b,b,b)$$

$$= (a+b,\ a+b,\ a+b) \in U$$

② $$kU = k(a,a,a)$$

$$= (ka, ka, ka) \in U$$

$\therefore$ U is a v.s of $\mathbb{R}^3$

Ex 2: Show that $W = \{ (a, b, c) \mid b = a + c \}$ ⓵ is a sub space of a v.s in $\mathbb{R}^3$

Soln: Take any $u, v \in W$, then

$u = (u_1, u_2, u_3)$ such that $u_2 = u_1 + u_3$ ⓶ from ⓵

$v = (v_1, v_2, v_3)$ " $v_2 = v_1 + v_3$ ⓷ from ⓵

clearly $W \neq \{\phi\}$ as $(0, 0, 0) \in W$ as $0 = 0 + 0$ from ⓵

⓵ $u + v = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3)$

$= (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

from ⓵ such that $u_2 + v_2 = (u_1 + u_3) + (v_1 + v_3)$ from ⓶ & ⓷

$= (u_1 + v_1) + (u_3 + v_3)$

$\Rightarrow u + v \in W$

(2) $kU = k(u_1, u_2, u_3)$

$= (ku_1, ku_2, ku_3)$

from ⓵ such that $ku_2 = ku_1 + ku_3$

$\Rightarrow ku \in W$

$\Rightarrow W$ is a sub space of $\mathbb{R}^3$

Ex 3: W is a set of singular matrix of order 2. Show that W is not a subspace of $M_{2\times 2}$ with the standard operations.
(Singular matrix $|A| = 0$)

Soln: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W$

since $|A| = 0$ & $|B| = 0$ $\therefore$ singular

Now

⓵ $A + B = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq 0 \notin W$

is not a subspace

Linear Combination: A vector v in a vector space V is called a linear combination of the vectors $u_1, u_2 \dots u_n$ in V. If v can be written in the form

$$v = c_1 u_1 + c_2 u_2 + \dots c_n u_n \qquad c_1, c_2 \dots c_n \text{ scalars}$$

Q. Finding a linear combination:

Let $u = (1, 2, -1)$ and $v = (6, 4, 2) \in \mathbb{R}^3$. Show that $w = (9, 2, 7)$ is a linear combination of u & v.

Sol: Suppose that

$$w = au + bv \quad \text{—} \ (1)$$

$$(9, 2, 7) = a(1, 2, -1) + b(6, 4, 2)$$

$$= (a, 2a, -a) + (6b, 4b, 2b)$$

$$= (a + 6b, 2a + 4b, -a + 2b)$$

on equating the corresponding components, we get

$$a + 6b = 9 \quad \text{—} \ (2)$$

$$2a + 4b = 2 \quad \text{—} \ (3)$$

$$-a + 2b = 7 \quad \text{—} \ (4)$$

on adding (2) & (4) we get

$$a - a + 6b + 2b = 9 + 7$$

$$0 + 8b = 16$$

$$b = \frac{16}{8} = 2$$

$$\boxed{b = 2}$$

using this value in (2)

$$a + 6 \times 2 = 9$$

$$a = 9 - 12 = -3$$

$$\boxed{a = -3}$$

Putting the values of a & b in (1)

$$w = -3u + 2v$$

[illegible] combination of u & v.