# Week-3, chapter 2

## Determinants

**Minors and Co-factors:** If $A$ is the square matrix

Then the minor of $a_{ij}$ is denoted by $M_{ij}$.

The number $(-1)^{i+j} M_{ij}$ is denoted by $C_{ij}$ is called the co-factor of entry $a_{ij}$.

Q1 Find the Minors and cofactor $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$

Soln: The minor of $a_{11}$ is $M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$

The co-factor of $a_{11}$ is

$$C_{11} = (-1)^{1+1} M_{11}$$
$$= (-1)^2 \times 16$$
$$= 16$$

Similarly, the minor of $a_{12}$

$$M_{12} = \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = 16 - 6 = 10$$

The co-factor of $a_{12}$ is

$$C_{12} = (-1)^{1+2} M_{12} =$$
$$= (-1)^3 \times 10$$
$$= -10$$

Similarly, the minor of $a_{32}$ is

$$M_{32} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

Co-factor of $a_{32}$ is $C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -26$ //

## co-factor expansion Along Row or Column wise

Q: Find the determinant by cofactor expansion along first Row Then first column.

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

Soln: By Row Side

(a) $\det(A) = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$

$= 3(-4) - (1)(-11) + 0 =$

$= -1$

(b) By column side

$= 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$

$= 3(-4) - (-2)(-2) + 5(3)$

$= -1$ //

Technique To evaluate $2\times 2$ or $3\times 3$ determinants.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right. \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array}$$

$= a_{11} \cdot a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} + a_{21} \cdot a_{32}$

$- a_{13} \cdot a_{22} \cdot a_{31} - a_{11} \cdot a_{23} \cdot a_{32} - a_{12} a_{21} a_{33}$

Q. Calculate $A = \begin{vmatrix} 1 & 5 & -3 \\ 1 & 0 & 2 \\ 3 & -1 & 2 \end{vmatrix}$

$$\left|\begin{array}{ccc|cc} 1 & 5 & -3 & 1 & 5 \\ 1 & 0 & 2 & 1 & 0 \\ 3 & -1 & 2 & 3 & -1 \end{array}\right.$$

Soln: we can write

$(-3)(0)(3) - (1)(2)(-1)$

Theorem 2.2.5 : If $A$ is a square matrix with two proportional rows or two proportional columns, Then $\det(A) = 0$

for ex

$$\begin{vmatrix} 1 & 3 & -2 & 4 \\ 2 & 6 & -4 & 8 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix}$$

$\therefore$ Second Row is 2 times The first Row

If we $R_2 \to R_2 - 2R_1$ we get $R_2 = 0$ $\therefore \det A = 0$

Evaluate the determinant by Row Reduction

In this we convert the determinant into upper triangular Matrix. using Row operation.

Q1 Evaluate $\operatorname{Det}(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix}$ $R_2 \leftrightarrow R_1$

Sol:

$$\operatorname{Det}(A) = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \quad R_1 \text{ take common } 3$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \quad R_3 \to R_3 - 2R_1$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \quad R_3 \to R_3 - 10R_2$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} \quad R_3 \text{ common } (-55)$$

$$(-3)(-55)\begin{vmatrix} 1 & -2 & 3 \end{vmatrix}$$

$= (-3)(-55)(1)$

$= 165$

Theorem 2.3.3 A square matrix $A$ is invertible iff $\det(A) \neq 0$

Q1 check weather $A$ is invertible or not $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$

Soln: $R_1$ and $R_3$ are proportional.

$\therefore \det(A) = 0$

$\therefore \det(A)$ is not invertable

Theorem: If $A$ and $B$ are square matrices of the same size.

Then $\det(AB) = \det(A)\det(B)$

Q. Consider $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}$,

$AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}$

Soln:

$\det(A) = 3 - 2 = 1$

$\det(B) = -8 - 15 = -23$

$\det(AB) = 28 - 51 = -23$

$\therefore \det(AB) = \det(A)\det(B)$

$-23 = 1 \times (-23)$ //

**Cramers Rule** : If $Ax = b$ is a system of $n$ linear equations in $n$ unknowns such that $\det(A) \neq 0$, Then the system has a unique solution, this solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \cdots \quad x_n = \frac{\det(x_n)}{\det(A)}$$

where

$$A_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad A_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$A_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Q. Using Cramers Rule to Solve

$$\begin{aligned} x_1 + \quad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

**Sol:** Given

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

Now $\det(A_1) = -40$, $\det(A) = 44$, $\det(A_2) = 72$

$\det(A_3) = 152$

Now find

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11} \quad //$$

Q If $A = \begin{pmatrix} 5 & -2 & 1 \\ 0 & 3 & -1 \\ 2 & 0 & 7 \end{pmatrix}$ and $|A| = 103$. Evaluate the determenants of the following matrices

$$B = \begin{pmatrix} 5 & -2 & 1 \\ 0 & 6 & -2 \\ 2 & 0 & 7 \end{pmatrix} \quad C = \begin{pmatrix} 5 & -2 & 1 \\ 2 & 0 & 7 \\ 0 & 3 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & -2 & 1 \\ 0 & 3 & -1 \\ 12 & -4 & 9 \end{pmatrix}$$

Soln: Theorem 2.2.3: Let $A$ be an $n \times n$ matrix

(a) If $B$ is the matrix that results when a single row or singe column of $A$ is multiplied by a scaler $k$, then $\det(B) = k \det(A)$

(b) If $B$ is the matrix that results when two rows or two columns of $A$ are interchanged, then $\det(B) = -\det(A)$

(c) If $B$ is the matrix that results when a multiple of one row of $A$ is adde to another row or when a multible of one column is added to anot column, then $\det(B) = \det(A)$

Soln: (B) $B$ is the matrix resulted when the second row $A$ is multiplied by 2. Now using Theorem (a)

$$|B| = 2|A| = 2 \cdot 103 = 206$$

(c) $C$ is the Matrix resulted when the second and third rows of $A$ are interchanged. Now using theorem (b)

$$|C| = -|A| = -103$$

(D) $D$ is the matrix resulted when the first row of $A$ multiplied by 2 and added to the third row of $A$. using Theorem (c)

$$|D| = |A| = 103$$

$$\dots = 17$$
$$3x + 2y + z = 11$$
$$x - 5y + z = -5$$

<u>Soln:</u>

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -5 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 17 \\ 11 \\ -5 \end{bmatrix}, \quad |A| = -48$$

$$A_1 = \begin{bmatrix} 17 & 2 & 3 \\ 11 & 2 & 1 \\ -5 & -5 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 17 & 3 \\ 3 & 11 & 1 \\ 1 & -5 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 2 & 17 \\ 3 & 2 & 11 \\ 1 & -5 & -5 \end{bmatrix}$$

Now we find that

$$|A_1| = -48, \quad |A_2| = \cancel{48} -96, \quad |A_3| = -192$$

Thus

$$x = \frac{|A_1|}{|A|} = \frac{-48}{-48} = 1$$

$$y = \frac{|A_2|}{|A|} = \frac{-96}{-48} = 2$$

$$z = \frac{|A_3|}{|A|} = \frac{-192}{-48} = 4$$

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