Week 2, chapter 1

Matrix Properties

(a) $A + B = B + A$ commutative law for addition

(b) $A + (B + C) = (A + B) + C$ (Associative law for add

(c) $A(BC) = (AB)C$ Associative Law for Multip

Note: In Matrix $AB \neq BA$

for ex: Consider $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

Then $AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}$ and $BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$

Thus $AB \neq BA$

Zero Matrix is denoted by 0 for ex:

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$ $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}_{1\times 3}$ etc $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$A + 0 = A$$
$$A + (-A) = 0$$
$$0(A) = 0$$

Identity matrix an $n \times n$ matrix with ones on the main diagonal and zeros elsewhere for eg:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Invertible (non singular) : If $AB = BA = I$, Then A is invertable for ex:

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus A and B are invertible and each is an inverse of the other.

Theorem 1.4.5 Inverse (old method)

If the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad - bc \neq 0$ Then the formula is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

for ex: (a) $A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

(a) The determinant $\det(A) = (6)(2) - (1)(5) = 7$ which is non-zero, Thus A is non-singular or invertable or we can solve it

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

(b) $\det(A) = (-1)(-6) - (2)(3) = 0$
we cant solve it.

Using the Row operation find the inverse (New Method)

Q1 Find the inverse of $A = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix}$

Sol: Now we can write

$$[A \mid I] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -2 & -7 & 0 & 1 \end{array}\right]$$

So the final matrix will be of the form

$[I \mid A^{-1}]$ Now lets start

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -2 & -7 & 0 & 1 \end{array}\right] \quad R_2 \to R_2 + 2R_1.$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right] \quad R_2 \to -R_2$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & -1 \end{array}\right] \quad R_1 \to R_1 - 3R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & 3 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

So we have

$$A = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 7 & 3 \\ -2 & -1 \end{bmatrix}$$

you can check by multiply each other we get the identity matrix. $A A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ex 4: Using Row operation E- find A^{-1}, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Soln:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & +1 & -3 & -2 & +1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \quad \cancel{R_1 \times -ve}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & +1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 + 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \quad \times -R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

... $n \times n$ matrix, then for each $n \times 1$ matrix b, the system of equations $Ax = b$ has exactly one solution, namely $X = A^{-1} b$.

Q. ~~find A^{-1}~~ Find the solution of system of linear equations using A^{-1}.

$$x + 3y = 1$$
$$2x + 5y = 3$$

Soln: we can write it like

$$A \quad X = b$$

coefficient matrix

variable matrix

constant matrix

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

so $\quad Ax = b$

or $\quad A^{-1}A \, X = A^{-1} b$

$$I X = A^{-1} b$$

or so $\boxed{X = A^{-1} b}$ —— (1)

Now

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$= \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \quad R_2 \times -ve$$

$$= \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \quad R_1 \to R_1 - 3R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$

Now from (1)

$$A^{-1} b = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

So from (1)

$x = 4$ and $y = -1$

Triangular Matrices

A square matrix in which all the entries above the main diagonal are zero is called lower triangular matrix. And a square matrix in which all the entries below the main diagonal are zero are called upper triangular matrix.

for ex:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}, \quad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

upper triangular Matrix   Lower triangular matrix