

Week-12, chapter 9

Numerical Methods

LU- Decomposition:

Step 1: Reduce the matrix to row Echelon form

Step 2: In each position along the main diagonal of L , Place the reciprocal of the multiplier

Step 3: In each position " " " " , Place the negative of the multiplier

Step 4: Form the decomposition $A = LU$

Q1 Find an LU-decomposition

$$A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$$

Sol: Given

$$A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \begin{matrix} R_1 \rightarrow \frac{R_1}{6} \\ \\ \end{matrix} \quad \begin{matrix} \times \text{ denote the unknown} \\ \text{entry of } I \end{matrix} \begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \begin{matrix} \text{multiplier} = \frac{1}{6} \\ R_2 \rightarrow R_2 - 9R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{matrix} \quad \begin{bmatrix} 6 & 0 & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 2 & 1 \\ 0 & 8 & 5 \end{bmatrix} \begin{matrix} \text{multiplier} = -9 \\ \text{multiplier} = -3 \end{matrix} \quad \begin{bmatrix} 6 & 0 & 0 \\ 9 & \times & 0 \\ 3 & \times & \times \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 8 & 5 \end{bmatrix} \text{ multiplier} = \frac{1}{2}$$

$$\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & \times & \times \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \text{ multiplier} = -8$$

$$\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & \times \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \text{ multiplier} = 1$$

$$L = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}$$

So we have constructed the LU-decomposition

$$A = LU = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = A$$

→ See Q₂ Next page.

Q₂ Power Method

Step 1: choose an arbitrary non-zero vector

Step 2: compute Ax_0 and multiply it by $\frac{1}{\max(Ax_0)}$

Step 3: compute Ax_1 and scale it by factor $\frac{1}{\max(Ax_1)}$

Similarly and so on

Q₁ Apply the power method with maximum entry scaling to $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ with $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ stop at x_5

Soln:

$$Ax_0 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and } x_1 = \frac{Ax_0}{\max(Ax_0)}$$

So $x_1 = \frac{1}{3} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \approx \begin{bmatrix} 1.0000 \\ 0.66667 \end{bmatrix}$

Now

$Ax_1 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1.0000 \\ 0.66667 \end{bmatrix} \approx \begin{bmatrix} 4.33333 \\ 4.00000 \end{bmatrix}$ and

$\lambda_2 = \frac{Ax_1}{\max(Ax_1)} \approx \frac{1}{4.33333} \begin{bmatrix} 4.33333 \\ 4.00000 \end{bmatrix} \approx \begin{bmatrix} 1.0000 \\ 0.92308 \end{bmatrix}$

$Ax_2 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1.0000 \\ 0.92308 \end{bmatrix} \approx \begin{bmatrix} 4.84615 \\ 4.76923 \end{bmatrix}$ and

$\lambda_3 = \frac{Ax_2}{\max(x_2)} = \frac{1}{4.84615} \begin{bmatrix} 4.84615 \\ 4.76923 \end{bmatrix} \approx \begin{bmatrix} 1.0000 \\ 0.98413 \end{bmatrix}$

$Ax_3 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1.0000 \\ 0.98413 \end{bmatrix} \approx \begin{bmatrix} 4.96825 \\ 4.95238 \end{bmatrix}$ and

$\lambda_4 = \frac{Ax_3}{\max(Ax_3)} = \frac{1}{4.96825} \begin{bmatrix} 4.96825 \\ 4.95238 \end{bmatrix} \approx \begin{bmatrix} 1.0000 \\ 0.99681 \end{bmatrix}$

$Ax_4 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1.0000 \\ 0.99681 \end{bmatrix} \approx \begin{bmatrix} 4.99361 \\ 4.99042 \end{bmatrix}$

$\lambda_5 = \frac{Ax_4}{\max(Ax_4)} = \frac{1}{4.99361} \begin{bmatrix} 4.99361 \\ 4.99042 \end{bmatrix} \approx \begin{bmatrix} 1.0000 \\ 0.99938 \end{bmatrix}$

Now

$\lambda_1 = \frac{(Ax_1)^T x_1}{x_1^T x_1} \approx \frac{7.00000}{1.44444} \approx 4.84615$

$\lambda_2 = \frac{(Ax_2)^T x_2}{x_2^T x_2} \approx \frac{9.24852}{1.05207} = 4.99361$

$\lambda_3 = \frac{(Ax_3)^T x_3}{x_3^T x_3} \approx \frac{9.84203}{1.96851} = 4.999974$

$\lambda_4 = \frac{(Ax_4)^T x_4}{x_4^T x_4} \approx \frac{9.96800}{1.99362} \approx 4.99999$

$$\lambda_5 = \frac{(Ax_5)^T x_5}{x_5^T x_5} \approx \frac{9.99360}{1.99872} \approx 5.00000.$$

Singular Value Decomposition: If A is $m \times n$ matrix and if $\lambda_1, \lambda_2, \dots$ are the E.V Eigen value of $A^T A$. Then the numbers $\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \dots$ are called the Singular values of A .

Q1 Find a Singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Soln: First find $A^T A$ is

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

~~The characteristic~~ Now to find the Eigen value of $A^T A$.

$$\text{The characteristic Eqn is } |\lambda I - A^T A| = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda) - 1 = 0$$

$$4 - 2\lambda - 2\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda = 3, 1$$

So $\lambda_1 = 3$ and $\lambda_2 = 1$ are the Eigen value so

The singular value of A is

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}, \quad \sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$$

Now find the E.V corresponding to $\lambda_1 = 3$ & $\lambda_2 = 1$ are

$$v_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Now to find u_i where $u_i = \frac{1}{\sigma_i} A v_i$

$$u_1 = \frac{1}{\sigma_1} A u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A u_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

with the help of u_1 and u_2 we can find u_3

$$u_3 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Now the singular value decomposition of A is

$$A = U \Sigma V^T$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 \\ \frac{\sqrt{6}}{6} & -\frac{1}{\sqrt{2}} \\ \frac{\sqrt{6}}{6} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Singular values are in diagonal

A_2 find an LU-decomposition of $A = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$

Sol:

$$A = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 + R_1} \begin{bmatrix} \times & 0 \\ \times & \times \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 + R_1} \begin{bmatrix} 2 & 0 \\ \times & \times \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \xrightarrow{R_2 \rightarrow \frac{R_2}{3}} \begin{bmatrix} 2 & 0 \\ -1 & \times \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \xrightarrow{\dots} \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$$

