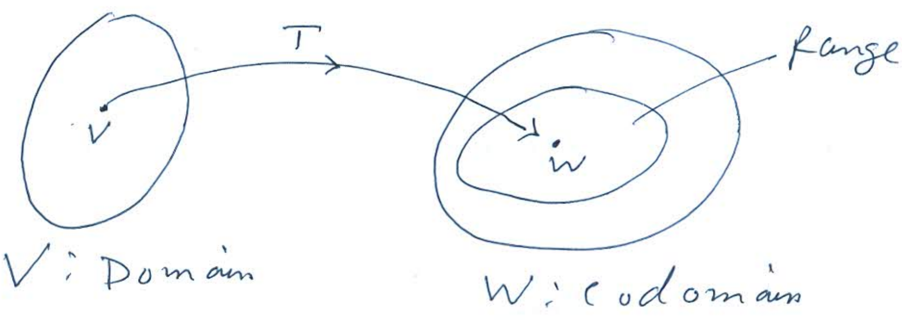
Week-11, chapter 8

Linear Transformations



$$T : V \to W$$

Function T that maps a vector space V into a vector space W:

Q1 $T: R^2 \to R^2$, $v = (v_1, v_2) \in R^2$, $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$ ← (1)

(a) Find the image of $v = (-1, 2)$ (b) Find the pre-image of $w = (-1, 11)$

Soln: (a) $v = (-1, 2)$

$$T(v) = T(-1, 2) = (-1-2, -1+2(2)) \quad \text{from (1)}$$
$$= (-3, 3)$$

(b) $T(v) = w = (-1, 11)$ $\quad v \xrightarrow{T} w$

we know $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$

So $v_1 - v_2 = -1$

So $v_1 - v_2 = -1$

$v_1 = 3$, $v_2 = 4$

...se of $(-1, 11)$

...ransformation: Let V, W be the Vector Space

$V \to W$: V to W linear Transformation if

$T(u+v) = T(u) + T(v) \quad \forall u, v \in V$

Linear T...

Q1 Varify a Linear Transformation T from R^2 into R^2

$$T(v_1, v_2) = (v_1 - v_2,\ v_1 + 2v_2) \quad — (1)$$

Sol: Let $u = (u_1, u_2)$ $v = (v_1, v_2)$ and c is any real No

1 - Let $u + v = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1,\ u_2 + v_2)$

so

~~T(uv~~

$$T(u+v) = T(u_1 + v_1,\ u_2 + v_2)$$

$$= ((u_1 + v_1) - (u_2 + v_2),\ (u_1 + v_1) + 2(u_2 + v_2))$$

from (1)

$$= ((u_1 - u_2) + (v_1 - v_2),\ (u_1 + 2u_2) + (v_1 + 2v_2))$$

$$= (u_1 - u_2,\ u_1 + 2u_2) + (v_1 - v_2,\ v_1 + 2v_2)$$

$$= T(u) + T(v) \quad \text{from (1)}$$

2 - Scalar Multiplication

$$cu = c(u_1, u_2) = (cu_1, cu_2)$$

Now

$$T(cu) = T(cu_1, cu_2)$$

$$= (cu_1 - cu_2,\ cu_1 + 2cu_2) \quad \text{from (1)}$$

$$= c(u_1 - u_2,\ u_1 + 2u_2)$$

$$= cT(u)$$

Therefor, T is a Linear Transformation.

Zero Transformation

$T : V \to W \qquad T(v) = 0 \qquad \forall\, v \in W$

Identity Transformation

$T : V \to V \qquad T(v) = v \qquad \forall\, v \in V$

Q1 Let $T : R^3 \to R^3$ be a L.T such that

$T(1,0,0) = (2,-1,4)$, $T(0,1,0) = (1,5,-2)$

$T(0,0,1) = (0,3,1)$ Find $T(2,3,-2)$

Sol: we can write

$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$

coz These are standard basis

Now Apply The Transformation both Sides

$T(2,3,-2) = 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1)$

$= 2(2,-1,4) + 3(1,5,-2) - 2(0,3,1)$

$= (7,7,0)$

Q2 The Function $T : R^2 \to R^3$ is defined by $T(v) = Av =$

$$\begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

(a) Find $T(v)$ where $v = (2,-1)$

Sol: $v = (2,-1)$

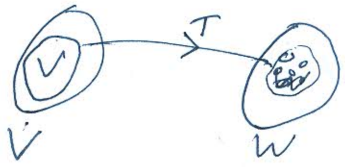
$$T(v) = Av = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \underset{R^2 \text{ vector}}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} = \underset{R^3 \text{ vector}}{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}$$

$\therefore T(2,-1) = (6,3,0)$

Kernel of a L.T: Let $T: V \to W$ be a L.T Then the set of all vectors v in V that satisfy $T(v) = 0$ is called the Kernal of T and it is denoted by ~~ket~~ ker (T)

$$\ker(T) = \{ v \mid T(v) = 0 \quad \forall v \in V \}$$

Note: Sometimes kernel T is called the Nullspace of T

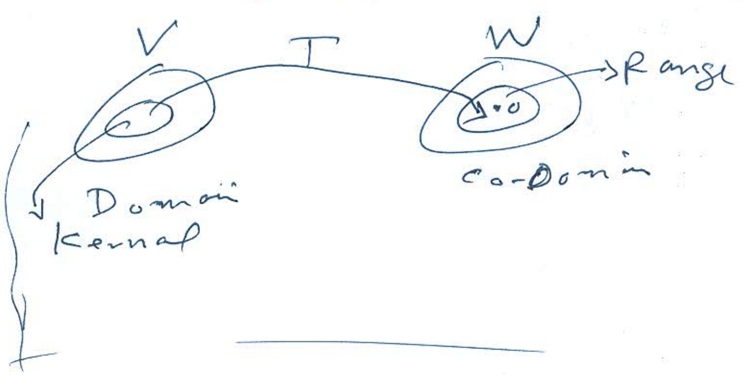


Range of a L.T: Let $T: V \to W$ be a L.T

Then the set of all vectors ~~in the~~ w in W that ~~a~~ are images of vectors in V is called the range of T and denoted by range (T)

Note: (1) Ker (T) is a subspace of V

(2) range (T) is a subspace of W



Rank of a L.T: Let $T: V \to W$

rank (T) = The dimension of the range of T

Nullity of L.T: Let $T: V \to W$

nullity (T) = The dimensions of the kernel of T

Dimension Theorem for a L.T: Let $T: V \to W$ be a L.T

rank (T) + nullity (T) = n

OR

dim (range of T) + dim (kert of T) = dim (domain of T)

Q. Find the rank and Nullity of L-T

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Soln: we know rank (T) = rank $(A) = 2$ no. of Non-zero Rows

and nullity (T) = dim (ker T)
= dim (domain of T)
= 3

and given $n = 3$ $\therefore$ matrix is 3×3

So by rank nullity Theorem

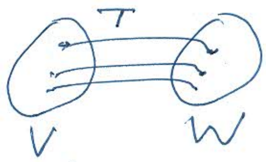
rank (T) + nullity of $T = n$

or nullity of $(T) = n -$ rank (T)

$= 3 - 2$

nullity $(T) = 1$

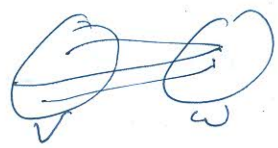
One-to-one: if $T(u) = T(v)$



one-to-one

If pre-image of every w in the range consist of a single vector

Onto: if ever element of W has a preimage in V



Theorem: Let $T: V \to W$ be L-T
Then T is 1-1 iff $\ker(T) = \{0\}$

Isomorphism: A L-T $T: V \to W$ That is 1-1 and onto is called isomorphism.

Matrix for L-T

(1) $T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$
OR

(2) $T(x) = Ax = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$