Week-10, chapter 7

Diagonalization of a Matrix

Orthogonal Matrix: A square matrix is said to be orthogonal if its transpose is same as its inverse ie

$$A^{-1} = A^T$$

$$\text{or} \quad A A^T = A^T A = I$$

Q1 check weather it is orthogonal $A = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$

Soln: Since its transpose is

$$A^T = \begin{bmatrix} 3/7 & -6/7 & 2/7 \\ 2/7 & 3/7 & 6/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix}$$

Now

$$A A^T = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Orthogonally Diagonalizing of $n \times n$ Symmetric Matrix

Step 1: Find the Eigen value

Step 2: Find the Eigen vectors corresponding to Eigen value

Step 3: Find P where $P = [\cancel{v_1, v_2, v_3}]\ [P_1, P_2, P_3]$

such that $P^T A P = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$

Q1 Find an orthogonal matrix P that diagonalise

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Soln: step 1: characteristic eqn is $|\lambda I - A| = \begin{vmatrix} \lambda-4 & -2 & -2 \\ -2 & \lambda-4 & -2 \\ -2 & -2 & \lambda-4 \end{vmatrix}$

$$= (\lambda-2)^2 (\lambda-8) = 0$$

$\lambda = 2, 2, 8$ are the Eigen values.

Now corresponding to $\lambda = 2, 2$ the Eigenvector are

$$v_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 2/\sqrt{6} \end{bmatrix}$$

and corresponding to $\lambda = 8$ the E.V is

$$v_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

<u>Step 3:</u> Now $P = [P_1, P_2, P_3]$ or we can write

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \quad \text{and} \quad P^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$P^T A P = \begin{bmatrix} & & \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} & & \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

<u>Quadratic Forms</u> A homogeneous Polynomial of second degree in any number of variables is called a quadratic form. for ex:

$$a_1 x_1^2 + a_2 x_2^2 + 2 a_3 x_1 x_2 \Rightarrow [x_1, x_2] \begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x^T A x$$

And

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + 2 a_4 x_1 x_2 + 2 a_5 x_1 x_3 + 2 a_6 x_2 x_3 \quad \text{Then}$$

$$[x_1 \; x_2 \; x_3] \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^T A x$$

Q1. Express the Quadratic Forms in Matrix Notation. (2)

(a) $2x^2 + 6xy - 5y^2$ (b) $x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 8x_2x_3$

Soln: (a)

$$2x^2 + 6xy - 5y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(b) $x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 8x_2x_3$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 4 \\ -1 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Theorem: If A is a Symmetric Matrix Then

(a) $x^T A x$ is +ve definite iff Eigen value of $A > 0$

(b) " -ve " $A < 0$

(c) " Indefinite iff atleast one +ve and one -ve

Q1. Find the nature of of the Quadratic forms

(a) $x^2 + 5y^2 + z^2 + 2xy + 6yz + \cancel{8}2zx$

(b) $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$

Soln: (a) The matrix of a Quadratic form is $x^T A x =$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

ie $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Eigen value are $-2, 3, 6$

So Since Eigen value are +ve & -ve So The given Quadratic form is Indefinite

(b) The matrix of a Quadratic form is $x^T A x =$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

ie $A = [\quad]$

The Eigen value are $2, 3, 6$

all these are +ve, so the given quadratic form is +ve definite.

Conjugate transpose: If A is complex matrix, Then

$$A^* = \bar{A}^T$$

Q. Find the Conjugate transpose A^* of $A = \begin{bmatrix} 1+i & -i & 0 \\ 2 & 3-2i & i \end{bmatrix}$

Soln:

$$\bar{A} = \begin{bmatrix} 1-i & i & 0 \\ 2 & 3+2i & -i \end{bmatrix} \quad \text{So}$$

$$A^* = \bar{A}^T = \begin{bmatrix} 1-i & 2 \\ i & 3+2i \\ 0 & -i \end{bmatrix}$$

Defn: (a) A square matrix is said to be unitary if

$A^{-1} = A^*$ or $A \cdot A^* = I$ and is said to Hermitian if

$$A^* = A$$

(b) The Eigen value of a Hermitian matrix are real

(c) The Eigen value of a Skew-Hermitian matrix are pure imaginary

(d) The Eigen value of a unitary matrix have absolute value 1.

Q. Find the Eigen value of Hermitian matrix $A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$

Soln: The characteristic eqn is

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & -1-i \\ -1+i & \lambda-3 \end{vmatrix}$$

$$= (\lambda-2)(\lambda-3) - (-1-i)(-1+i)$$

$$= (\lambda^2 - 5\lambda + 6) - 2 = (\lambda-1)(\lambda-4)$$

$\lambda = 1, 4$ which are real

So from (a) The Eigen value of an Hermitian matrix are real.

Q Show that $A = \begin{bmatrix} 7 & 1+i & 8 \\ 1-i & 5 & -1-6i \\ 8 & 6i-1 & -1 \end{bmatrix}$ is Hermitian

Soln: we know to show hermitian

$A^* = \bar{A}^T$

$$\bar{A} = \begin{bmatrix} 7 & 1-i & 8 \\ 1+i & 5 & -1+6i \\ 8 & -6i-1 & -1 \end{bmatrix}$$

$$A^* = \begin{bmatrix} 7 & 1+i & 8 \\ 1-i & 5 & -1-6i \\ 8 & 6i-1 & -1 \end{bmatrix} = A$$

Q5 Is the matrix $A = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}$ orthogonal

Sin $A A^T = \begin{pmatrix} \ \end{pmatrix} \begin{pmatrix} \ \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$

Q6. Express the quadratic form in the matrix notation $x^T A x$ where A is a symmetric matrix

(a) $3x_1^2 + 7x_2^2$ (b) $9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + x_2x_3$

Soln: (a) $[x_1 \ x_2] \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(b) $[x_1 \ x_2 \ x_3] \begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & 1/2 \\ -4 & 1/2 & 4 \end{bmatrix}$

Q7. Show that the matrix $A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$ is unitary

Soln: we know that a square unitary matrix A is said to be unitary if $A^*.A = I$ where A^* is a conjugate transpose of A.

$$A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \quad \text{so } A^* = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$