Week-9, chapter 6

Inner Product Spaces

Defn: Let u, v, w be vectors in a vector space V, and let c be any scalar; An inner product on V is a function that associate a real number $\langle u, v\rangle$ with each pair of vectors u & v and satisfies the following axioms.

1 - $\langle u, v\rangle = \langle v, u\rangle$

2 - $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$

3 - $c\langle u, v\rangle = \langle cu, v\rangle$

4 - $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ iff $v = 0$

Note: A vector space V i.e. $(V, +, \cdot)$ with an inner product is called an inner product space $(V, +, \cdot, \langle , \rangle)$

Q1 Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be vectors in R^2, verify that the Euclidean inner product $\langle u, v\rangle = 3u_1v_1 + 2u_2v_2$ satisfies the four inner product axioms

Soln: Axiom 1: $\langle u, v\rangle = 3u_1v_1 + 2u_2v_2$

$= 3v_1u_1 + 2v_2u_2$

$= \langle v, u\rangle$

Axiom 2: If $w = (w_1, w_2)$ Then

$\langle u+v, w\rangle = 3(u_1+v_1)w_1 + 2(u_2+v_2)w_2$

$= 3(u_1w_1 + v_1w_1) + 2(u_2w_2 + v_2w_2)$

$= (3u_1w_1 + 2u_2w_2) + (3v_1w_1 + 2v_2w_2)$

$= (u, w) + (v, w)$

Axiom 3: $\langle cu, v\rangle = 3(cu_1)v_1 + 2(cu_2)v_2$

Axiom 3: $\langle cu, v\rangle = 3(cu_1)v_1 + 2($

$\underline{\text{Axiom 4}}$: $\langle v, v\rangle = 3(v_1 v_1) + 2(v_2 v_2)$

$= 3v_1^2 + 2v_2^2 \geq 0$

with equality iff $v_1 = v_2 = 0$

Q2. Show that the function define an inner product in R^2
where $u = (u_1, u_2)$ and $v = (v_1, v_2)$

$\langle u, v\rangle = u_1 v_1 + 2u_2 v_2$

$\underline{\text{Sol}}$: $\underline{\text{axiom 1}}$: $u_1 v_1 + 2u_2 v_2 =$

$= v_1 u_1 + 2v_2 u_2$

$= \langle v, u\rangle$

$\underline{\text{axiom 2}}$: $w = (w_1, w_2)$

$\langle u, v+w\rangle = u_1(v_1 + w_1) + 2u_2(v_2 + w_2)$

$= u_1 v_1 + u_1 w_1 + 2u_2 v_2 + 2u_2 w_2$

$= (u_1 v_1 + 2u_2 v_2) + (u_1 w_1 + 2u_2 w_2)$

$= \langle u, v\rangle + \langle u, w\rangle$

$\underline{\text{axiom 3}}$: $c\langle u, v\rangle = c(u_1 v_1 + 2u_2 v_2)$

$= (cu_1)v_1 + 2(cu_2)v_2$

$= \langle cu, v\rangle$

$\underline{\text{axiom 4}}$: $\langle v, v\rangle = v_1^2 + 2v_2^2 \geq 0$

$\langle v, v\rangle = 0 \Rightarrow v_1^2 + 2v^2 = 0 \Rightarrow v_1 = v_2 = 0$

Properties of inner Product

1- $\langle 0, v\rangle = \langle v, 0\rangle = 0$

2- $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

3- $\langle u, cv\rangle = c\langle u, v\rangle$

4- Norm (length of u), $\|u\| = \sqrt{\langle u, u\rangle}$ $\quad \therefore \|u\|^2 = \langle u$

stance b/w u and v

$$d(u,v) = \|u-v\| = \sqrt{\langle u-v,\, u-v\rangle}$$

Angle b/w two non-zero vectors u & v

$$\cos\theta = \frac{\langle u,v\rangle}{\|u\|\,\|v\|}, \quad 0\le\theta\le\pi$$

Note if $u \perp v$ Then $\langle u,v\rangle = 0$ othogonal

if $\|v\| = 1$ Then v is called the unit vectors

Q calculating the inner products $\langle u-2v,\, 3u+4v\rangle$

Sol: $\langle u-2v, 3u+4v\rangle = \langle u, 3u+4v\rangle - \langle 2v, 3u+4v\rangle$

$= \langle u,3u\rangle + \langle u,4v\rangle - \langle 2v,3u\rangle - \langle 2v,4v\rangle$

$= 3\langle u,u\rangle + 4\langle u,v\rangle - 6\langle v,u\rangle - 8\langle v,v\rangle$

$= 3\|u\|^2 + 4\langle u,v\rangle - 6\langle u,v\rangle - 8\|v\|^2$

$= 3\|u\|^2 - 2\langle u,v\rangle - 8\|v\|^2$

Later

Q2. consider the vectors $u = (2+3i, -1+5i)$, $v = (1+i, -i)$

or $v = (1-i, i)$

compute

(a) $\langle u,v\rangle$ and show that u & v are orthogonal

(b) $\|u\|$ and $\|v\|$ (c) $d(u,v)$

$u = (2+3i, -1+5i)$

$v = (1-i, i)$

$\langle u,v\rangle = (2+3i)(1-i) + (-1+5i)(i)$

Sol: (a) $\langle u,v\rangle = (2+3i)(1-i) + (-1+5i)(i)$

$= 2-2i+3i+3-i-5$

$= 5+i-i-5$

$= 0$ ∴ u & v are orthogonal

(b) $\|u\| = \sqrt{(2+3i)(2-3i) + (-1+5i)(-1-5i)}$

$= \sqrt{13+26} = \sqrt{39}$

$\|v\| = \sqrt{(1+i)(1-i) + (-i)(i)}$

$= \sqrt{3}$

(c) $d(u,v) = \|u-v\| = \|(2+3i, -1+5i) - (1+i, -i)\|$

$= \|(1+2i, -1+6i)\|$

$= \sqrt{(1+2i)(1-2i) + (-1+6i)(-1-6i)} = \sqrt{5+37}$

$= \sqrt{42}$

Q3. Show that the following set is an orthonormal basi

$S = \{ (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\}, \{-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}), (\frac{2}{3}$

v_1 $\qquad v_2 \qquad v_3$

Sol: first we show these vectors are orthogonal Then we show These vectors are normal, then we say the vectors are orthonormal.

$v_1 \cdot v_2 = \frac{1}{\sqrt{2}} \cdot (\frac{-\sqrt{2}}{6}) + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{6} + 0 \times \frac{2\sqrt{2}}{3}$

$= -\frac{1}{6} + \frac{1}{6} + 0$

$= 0$

$v_1 \cdot v_3 = \frac{1}{\sqrt{2}} \times \frac{2}{3} + \frac{1}{\sqrt{2}} \times -\frac{2}{3} + 0 \times \frac{1}{3}$

$= \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0$

$= 0$

$v_2 \cdot v_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$

Now for mal

$\|v_1\| = \sqrt{v_1 \cdot v_1} = \sqrt{(\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 + (0)^2}$

$= \sqrt{\frac{1}{2} + \frac{1}{2} + 0}$

$= \sqrt{1} = 1$

$\|v_2\| = \sqrt{v_2 \cdot v_2} = \sqrt{\frac{2}{36} + \frac{2}{36} + \frac{8}{4}} = 1$

$\|v_3\| = \sqrt{v_3 \cdot v_3} = \sqrt{\frac{4}{9} + \frac{4}{4} + \frac{1}{9}} = 1$

Thus S is an orthonormal.