21

Linear Programming 2: Basic Concepts

The basic concepts needed to develop the simplex method for solving linear programming problems are presented.

PREREQUISITES: Linear Systems
Matrices
Linear independence
Euclidean space R^n
Chapter 20: Linear Programming 1

INTRODUCTION

In the last chapter we presented a geometric technique for solving linear programming problems in two variables. However, this technique is not practical for the solution of linear programming problems in three or more variables. In this chapter we develop the fundamental ideas behind an algebraic technique, called the *simplex method,* for solving linear programming problems in any number of variables. The simplex method itself is presented in Chapter 22.

The *general linear programming problem in n variables*, described below, is analogous to Problem 20.1 in the last chapter.

PROBLEM 21.1 *Find values of* $x_1, x_2, \ldots, x_n$ *which either maximize or minimize*

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{21.1}$$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \;(\le)(\ge)(=)\; b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \;(\le)(\ge)(=)\; b_2 \\ \vdots \qquad\qquad \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \;(\le)(\ge)(=)\; b_m \end{aligned} \tag{21.2}$$

and

$$x_i \ge 0, \qquad \text{for } i = 1, 2, \ldots, n. \tag{21.3}$$

As in Chapter 20, the linear function z in (21.1) is called the *objective function* and conditions (21.2) and (21.3) are called the *constraints* of the problem.

For our purposes, it is necessary that we first convert our general linear programming problem to the following *standard form*:

PROBLEM 21.2 *Find values of* $x_1, x_2, \ldots, x_n$ *which maximize*

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{21.4}$$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{aligned} \tag{21.5}$$

and

$$x_i \ge 0, \qquad \text{for } i = 1, 2, \ldots, n. \tag{21.6}$$

Any linear programming problem can always be put in this standard form using the following three steps:

Step 1. Convert a minimization problem to a maximization problem by defining a new objective function

$$z' = -z.$$

For example, the problem of minimizing the objective function

$$z = 2x_1 + 3x_2 - 5x_3$$

is equivalent to the problem of maximizing the objective function

$$z' = -2x_1 - 3x_2 + 5x_3 .$$

Step 2. Convert a $\geq$ constraint to a $\leq$ constraint by multiplying the inequality by -1. Thus, the constraint

$$-x_1 + 2x_2 - 4x_3 \geq 6$$

is the same as the constraint

$$x_1 - 2x_2 + 4x_3 \leq -6.$$

Step 3. Convert a $\leq$ constraint to an equality constraint by adding a nonnegative *slack variable* to the lefthand side of the inequality. For example, if the original problem contains three variables, and one of the constraints is

$$x_1 - 2x_2 + 4x_3 \leq -6,$$

we add a new variable $x_4 \geq 0$ to the lefthand side to obtain

$$x_1 - 2x_2 + 4x_3 + x_4 = -6.$$

The variable x_4 takes up the slack between the two sides of the inequality. In this way, a new variable is added to the problem for each $\leq$ constraint. We assign each slack variable introduced a coefficient $c_i = 0$ in the objective function, so that the objective function is not affected by the values of the slack variables.

EXAMPLE 21.1 Convert the following linear programming problem to one in standard form:

$$\text{Minimize} \quad z = 3x_1 - 2x_2 + x_3 - x_4$$

Subject to

$$2x_1 + 5x_2 - 6x_3 - x_4 \leq 2$$
$$x_1 - 7x_2 - 5x_3 + 2x_4 \geq 6$$
$$2x_1 - 8x_2 - 8x_3 + 6x_4 = 5$$

and

$$x_1, x_2, x_3, x_4 \geq 0.$$

SOLUTION The first step is to multiply the objective function by -1 to convert the problem to a maximization problem. The second step is to multiply the second constraint by -1 to convert it to a $\leq$ inequality. The third step is to add slack variables x_5 and x_6 to the first and second constraints to convert them to equalities. The final problem in standard form is

$$\text{Maximize} \quad z' = -3x_1 + 2x_2 - x_3 + x_4 + 0x_5 + 0x_6$$

subject to

$$2x_1 + 5x_2 - 6x_3 - x_4 + x_5 = 2$$
$$-x_1 + 7x_2 + 5x_3 - 2x_4 + x_6 = -6$$
$$2x_1 - 8x_2 - 8x_3 + 6x_4 = 5$$

and

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

It will be convenient to use matrix notation to express Problem 21.2 in a more compact form as follows (the expression $\mathbf{x} \geq 0$ below denotes that each entry of the vector $\mathbf{x}$ is nonnegative):

PROBLEM 21.2 (in matrix notation)

$$\textit{Maximize} \quad z = \mathbf{c}^t\mathbf{x}$$

subject to

$$A\mathbf{x} = \mathbf{b}$$

and

$$\mathbf{x} \geq 0,$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_m & a_m & \cdots & a_{mn} \end{bmatrix}.$$

In this formulation, the problem is to find a nonnegative vector $\mathbf{x}$ in the vector space R^n which satisfies the constraint condition $A\mathbf{x} = \mathbf{b}$ and which makes the objective function $z = \mathbf{c}^t\mathbf{x}$ as large as possible. Analogous to the terminology in Chapter 20, any nonnegative vector $\mathbf{x}$ which satisfies the constraint $A\mathbf{x} = \mathbf{b}$ is called a *feasible solution* to the linear programming problem. The set of all feasible solutions in R^n is called the *feasible set* or the *feasible region* of the problem. A feasible solution which maximizes the objective function is called an *optimal solution.* As in Chapter 20, there are three possible outcomes to a linear programming problem:

(i) The constraints are inconsistent so that there are no feasible solutions.

(ii) The feasible set is unbounded and the objective function can be made arbitrarily large.

(iii) There is at least one optimal solution.

In most realistic applications only case (iii) arises.

We now examine the nature of the feasible set of a linear programming problem. To begin, let us introduce the following definition:

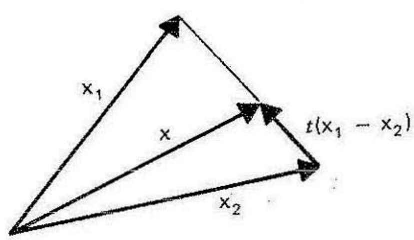
DEFINITION 21.1 *A set of vectors in* R^n *is called* convex *if whenever* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *belong to the set, so does the vector*

$$\mathbf{x} = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2$$

for any number t *in the interval* $[0, 1]$.

Geometrically, the vector $x = tx_1 + (1 - t)x_2$ lies on the line segment connecting the tips of the vectors x_1 and x_2 (Fig. 21.1). Thus, a convex set can be viewed as one in which the line segment connecting any two points in the set also belongs to the set. Figures 21.2(a), (b), and (c) illustrate three convex sets in R^2. Figure 21.1(d) is an example of a set in R^2 which is not convex. In Exercise 21.7, we ask the reader to prove the following theorem:

> **THEOREM 21.1** *The feasible set of a standard linear programming problem is convex.*



$$x = x_2 + t(x_1 - x_2)$$
$$= tx_1 + (1 - t)x_2$$

Figure 21.1

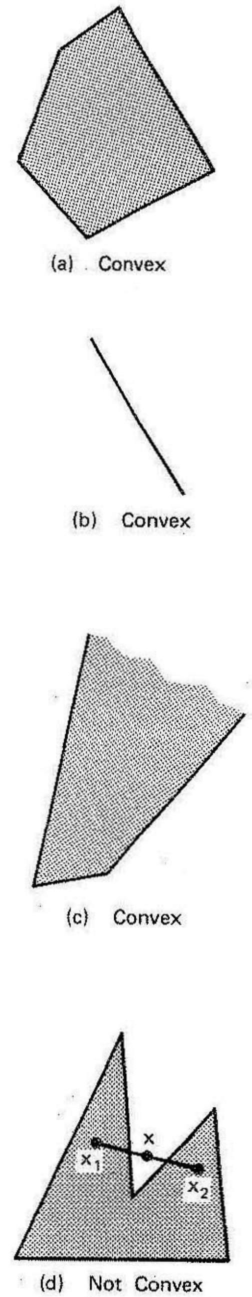


Figure 21.2

EXAMPLE 21.2 For the following linear programming problem:

$$\text{Maximize} \quad z = 3x_1 + x_2 - 2x_3$$

subject to

$$3x_1 + x_2 + x_3 = 10$$
$$2x_1 - x_2 + 2x_3 = 10$$

and

$$x_1, x_2, x_3 \geq 0,$$

the feasible set consists of the portion of the intersection of the two planes

$$3x_1 + x_2 + x_3 = 10$$
$$2x_1 - x_2 + 2x_3 = 10$$

which lies in the first octant of $x_1x_2x_3$-space. As we ask the reader to show in Exercise 21.8, this intersection consists of the line segment connecting the points

(5/2, 0, 5/2) and (0, 10/3, 20/3)

(Fig. 21.3), which is clearly convex.

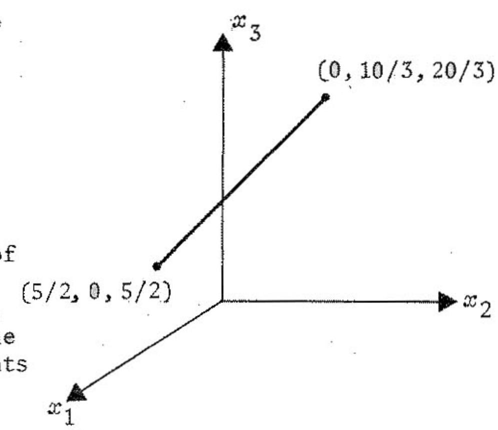


Figure 21.3

EXAMPLE 21.3 For the following linear programming problem:

$$\text{Maximize} \quad z = x_1 - 3x_2 + 2x_3$$

subject to

$$2x_1 + 4x_2 + 3x_3 = 12$$

and

$$x_1, x_2, x_3 \geq 0,$$

the feasible set consists of the portion of the plane

$$2x_1 + 4x_2 + 3x_3 = 12$$

which lies in the first octant of $x_1x_2x_3$-space. The reader can easily verify that Fig. 21.4 is a diagram of this feasible set and that it is convex.

In Chapter 20, we introduced the concept of an extreme point of a feasible set in x_1x_2-space. For an arbitrary convex set in R^n, the corresponding definition is the following:

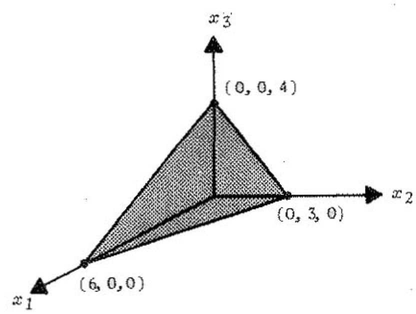


Figure 21.4

DEFINITION 21.2 *A vector* **x** *in a convex set is an* **extreme point** *of the convex set if*

$$\mathbf{x} \neq \tfrac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$$

for any two vectors $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *in the set.*

In other words, the extreme points of a convex set are those points which do not lie midway between any two points in the set. For example, the extreme points of the convex set illustrated in Fig. 21.2 are the two endpoints

$$(5/2,\ 0,\ 5/2) \quad \text{and} \quad (0,\ 10/3,\ 20/3)$$

of the straight-line segment. For the convex set illustrated in Fig. 21.4, the extreme points are the three corner points

$$(6,\ 0,\ 0), \quad (0,\ 3,\ 0), \quad \text{and} \quad (0,\ 0,\ 4)$$

of the triangular-shaped region.

Recall that in the last chapter we called a set in R^2 bounded if it can be enclosed in a sufficiently large circle; that is, if there is some positive radius r such that each point $\mathbf{x} = (x_1, x_2)$ in the set satisfies

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2} \leq r .$$

Similarly, we can define a bounded set in R^n as follows:

DEFINITION 21.3 *A set in R^n is said to be* **bounded** *if there is some positive number r such that each point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in the set satisfies*

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \le r.$$

The following theorem, which is just Theorem 20.1 of Chapter 20 extended to the space R^n, shows that if a linear programming problem has an optimal solution, it can be found among the extreme points of the feasible set.

THEOREM 21.2 *If the feasible set of a linear programming problem is nonempty and bounded, then the objective function attains its maximum at an extreme point of the set. If the feasible set is unbounded, then the objective function may or may not attain a maximum value; however, if it attains a maximum value, it does so at an extreme point.*

The proof of this theorem follows the same lines as that outlined in Chapter 20 for two variables. Let us apply this theorem to the two problems posed in Examples 21.2 and 21.3.

EXAMPLE 21.2 (REVISITED) As illustrated in Fig. 21.3, the feasible set for this problem is nonempty and bounded. Consequently, by Theorem 21.2 the objective function

$$z = 3x_1 + x_2 - 2x_3$$

attains its maximum at one of the two extreme points of the set. At the extreme point

$$(5/2,\ 0,\ 5/2)$$

we have

$$z = 5/2,$$

and at the extreme point

$$(0,\ 10/3,\ 20/3)$$

we have

$$z = -10.$$

Thus, the maximum value of the objective function is $z = 5/2$, and an optimal solution to the problem is $x_1 = 5/2$, $x_2 = 0$, and $x_3 = 5/2$.

EXAMPLE 21.3 (REVISITED) The feasible set of Example 21.3 is illustrated in Fig. 21.4. Since it is nonempty and bounded, Theorem 21.2 guarantees that the objective function

$$z = x_1 - 3x_2 + 2x_3$$

attains its maximum value at one of the three extreme points. The following table gives the values of the objective function at these extreme points:

Extreme point (x_1, x_2, x_3)	Value of $z = x_1 - 3x_2 + 2x_3$
(6, 0, 0)	6
(0, 3, 0)	-9
(0, 0, 4)	8

Thus the maximum value of z is 8 and an optimal solution is $x_1 = 0$, $x_2 = 0$, $x_3 = 4$.

In the above two examples (and in all of the examples in Chapter 20) the extreme points of the feasible sets were found geometrically. But geometric techniques are not possible if the problem has more than three variables. In such cases, we need an algebraic technique for generating the extreme points of the feasible set. In the next section we describe such an algebraic technique.

BASIC FEASIBLE SOLUTIONS

Let us reexamine the linear system

$$A\mathbf{x} = \mathbf{b} \tag{21.7}$$

of Problem 21.2, where A is an $m \times n$ matrix. Although it is not essential, we will assume for simplicity that $m \leq n$; i.e., that there are no more constraints than variables. We shall also assume that the m rows of A are linearly independent. This implies that A contains m linearly independent columns since the row space and column space of any matrix have the same dimension. In Exercise 21.10, we ask the reader to show that the linear system (21.7) can also be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{21.8}$$

where $\mathbf{a}_i$ ($i = 1, 2, \ldots, n$) is the i-th column vector of the matrix A. Thus, solving $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ is equivalent to solving the vector equation (21.8) for $x_1, x_2, \ldots, x_n$. For convenience, we say that the variable x_i "corresponds" to the column vector $\mathbf{a}_i$. As noted above, there must exist at least one set of m linearly independent vectors among the column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. For example, suppose the first m of these vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$, form such a set. Since these m vectors lie in R^m, and since R^m is m-dimensional, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ constitute a basis for R^m. Thus the vector $\mathbf{b}$ in (21.8) is uniquely expressible as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$. That is, there is a unique solution of (21.8) for which

$$x_{m+1} = x_{m+2} = \cdots = x_n = 0. \tag{21.9}$$

Similarly, any set of m linearly independent column vectors of A would lead to a solution of (21.8) in which $n - m$ of the variables are zero and the remaining m are uniquely determined. This suggests the following definition:

DEFINITION 21.4 *A vector*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is called a **basic solution** *of the linear system $A\mathbf{x} = \mathbf{b}$ if $n - m$ of the variables $x_1, x_2, \ldots, x_n$ are zero and the remaining m variables correspond to linearly independent column vectors of A. The $n - m$ zero variables are called the* **nonbasic variables**, *and the m variables corresponding to the linearly independent column vectors are called the* **basic variables** *of* $\mathbf{x}$.

A linear system of m equations in n variables has as many basic solutions as there are sets of m linearly independent columns among the n columns of the coefficient matrix A.

EXAMPLE 21.4 Find all basic solutions of the linear system

$$\begin{aligned} 2x_1 \quad\;\; + x_3 + 4x_4 + 2x_5 &= 20 \\ x_1 + x_2 - x_3 + x_4 + 3x_5 &= 10. \end{aligned}$$

SOLUTION The coefficient matrix of the linear system is

$$A = \begin{bmatrix} 2 & 0 & 1 & 4 & 2 \\ 1 & 1 & -1 & 1 & 3 \end{bmatrix}.$$

As we can see, any two columns of this matrix are linearly independent. Thus to find a basic solution we choose any two of these columns, set the three appropriate nonbasic variables equal to zero and solve the resulting 2×2 linear system for the two basic variables. For example, if we choose the two columns

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

we set the nonbasic variables

$$x_2,\ x_4,\ \text{and } x_5$$

equal to zero and obtain the system

$$\begin{aligned} 2x_1 + x_3 &= 20 \\ x_1 - x_3 &= 10 \end{aligned}$$

for the basic variables x_1 and x_3. This system is easily found to have the solution $x_1 = 10$ and $x_3 = 0$. The resulting basic solution to the original problem is

$$\mathbf{x} = \begin{bmatrix} \underline{10} \\ 0 \\ \underline{0} \\ 0 \\ 0 \end{bmatrix},$$

where we have underlined the basic variables. As the reader can verify, the ten possible pairs of columns of A lead to the following ten basic solutions:

$$\begin{bmatrix} \underline{10} \\ \underline{0} \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \underline{10} \\ 0 \\ \underline{0} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \underline{10} \\ 0 \\ 0 \\ \underline{0} \\ 0 \end{bmatrix} \begin{bmatrix} \underline{10} \\ 0 \\ 0 \\ 0 \\ \underline{0} \end{bmatrix} \begin{bmatrix} 0 \\ \underline{30} \\ \underline{20} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \underline{5} \\ 0 \\ \underline{5} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \underline{-20} \\ 0 \\ 0 \\ \underline{10} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \underline{-4} \\ \underline{6} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \underline{8} \\ 0 \\ \underline{6} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \underline{4} \\ \underline{2} \end{bmatrix} \quad (21.10)$$

where we have underlined the two basic variables in each case.

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From (21.10), we see that the first four basic solutions are equal as vectors in the vector space R^5. Nevertheless, we shall consider them to be distinct basic solutions because they result from different pairs of linearly independent columns of A. This particular circumstance arises because one of the basic variables in each of these four basic solutions is equal to zero. In general, a basic solution is said to be *degenerate* if any of its basic variables is equal to zero. Otherwise, it is said to be *nondegenerate*.

Recall that the feasible set of a standard linear programming problem consists of those vectors which satisfy a linear equation

$$A\mathbf{x} = \mathbf{b}$$

and which satisfy the nonnegativity condition

$$\mathbf{x} \geq 0.$$

Adjoining the nonnegativity condition to the concept of a basic solution we are led to the following definition:

DEFINITION 21.5 *In a standard linear programming problem a feasible solution which is also a basic solution of the system* $A\mathbf{x} = \mathbf{b}$ *is called a* **basic feasible solution.**

We are now ready to state the following fundamental theorem in the theory of linear programming:

THEOREM 21.3 *A vector* $\mathbf{x}$ *is an extreme point of the feasible set of a linear programming problem if and only if it is a basic feasible solution of the problem.*

This theorem will yield an algebraic technique for finding extreme points. The proof of this theorem is too long to present here. Instead, we refer the reader to any standard text in linear programming theory, such as S.I. Gass, *Linear Programming*, 4th ed., New York: McGraw-Hill Book Company, 1975.

EXAMPLE 21.5 Find an optimal solution of the linear programming problem:

$$\text{Maximize} \quad z = 2x_1 + 3x_2 - x_3 + 4x_5 + x_6$$

subject to

$$\begin{aligned} 2x_1 \phantom{{}+x_2} + x_3 + 4x_4 + 2x_5 &= 20 \\ x_1 + x_2 - x_3 + x_4 + 3x_5 &= 10 \end{aligned}$$

and

$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

SOLUTION As we ask the reader to show in Exercise 21.9, the feasible set of this problem is bounded. Consequently, from Theorem 21.2 the objective function attains its maximum value at an extreme point. The linear system $A\mathbf{x} = \mathbf{b}$ in the constraints is the same system considered in Example 21.4. Equations (21.10) give the ten basic solutions of this linear system. Of these ten, the eight listed in (21.11) are basic feasible solutions since they satisfy the nonnegativity condition. Thus, from Theorem 21.3, the extreme points of the feasible set are also given by Eq. (21.11). (Notice that these eight basic feasible solutions determine only five distinct extreme points.) Below each of the eight basic feasible solutions we have given the corresponding value of the objective function. From this we see that $z = 70$ is the maximum value of the objective function and an optimal solution is $x_1 = 0$, $x_2 = 30$, $x_3 = 20$, $x_4 = 0$, $x_5 = 0$.

$$\underset{z=20}{\mathbf{x}_1 = \begin{bmatrix} \underline{10} \\ \underline{0} \\ 0 \\ 0 \\ 0 \end{bmatrix}} \quad \underset{z=20}{\mathbf{x}_2 = \begin{bmatrix} \underline{10} \\ 0 \\ \underline{0} \\ 0 \\ 0 \end{bmatrix}} \quad \underset{z=20}{\mathbf{x}_3 = \begin{bmatrix} \underline{10} \\ 0 \\ 0 \\ \underline{0} \\ 0 \end{bmatrix}} \quad \underset{z=20}{\mathbf{x}_4 = \begin{bmatrix} \underline{10} \\ 0 \\ 0 \\ 0 \\ \underline{0} \end{bmatrix}} \quad \underset{z=70}{\mathbf{x}_5 = \begin{bmatrix} 0 \\ \underline{30} \\ \underline{20} \\ 0 \\ 0 \end{bmatrix}} \quad \underset{z=35}{\mathbf{x}_6 = \begin{bmatrix} 0 \\ \underline{5} \\ 0 \\ \underline{5} \\ 0 \end{bmatrix}} \quad \underset{z=-2}{\mathbf{x}_7 = \begin{bmatrix} 0 \\ 0 \\ \underline{8} \\ 0 \\ \underline{6} \end{bmatrix}} \quad \underset{z=18}{\mathbf{x}_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \underline{4} \\ \underline{2} \end{bmatrix}} \qquad (21.11)$$

The technique used in Example 21.5 can, in principle, be used to solve any linear programming problem. However, the number of basic feasible solutions quickly increases as the number of variables increases. For example, a linear programming problem with forty variables and twenty equality constraints could have over 130 billion basic feasible solutions. It would be completely impractical to find all of them, even with the fastest computer. Chapter 21 describes a practical alternative to this technique called the *simplex method*. We finish this chapter with a brief qualitative description of the simplex method to prepare for that chapter.

INTRODUCTION TO THE SIMPLEX METHOD

Let us introduce the following definition:

> **DEFINITION 21.6** *In a linear programming problem, two basic feasible solutions having m basic variables are said to be* **adjacent** *if they have $m - 1$ basic variables in common.*

Geometrically, extreme points corresponding to adjacent basic feasible solutions are connected by some "edge" of the feasible set. However, we shall not pursue this geometric interpretation.

EXAMPLE 21.5 (REVISITED) Let us find the adjacent basic feasible solutions of the linear programming problem posed in Example 21.5. The eight basic feasible solutions of this problem are given in Eq. (21.11). In Fig. 21.5 we have drawn a graph of the eight basic feasible solutions. In this graph we have linked adjacent basic feasible solutions with a single line. For example, x_5 and x_7 are adjacent because they have $m - 1 = 2 - 1 = 1$ basic variable in common, namely x_3. On the other hand, x_4 and x_5 are not adjacent since they do not have $m - 1 = 1$ basic variable in common.

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The simplex method is a way of proceeding from one basic feasible solution to an adjacent basic feasible solution in such a way that the value of the objective function never decreases. This usually leads to a basic feasible solution for which the value of the objective function is as large as possible. We say "usually" because there is a slight complication caused by degeneracy which we shall describe below.

In Fig. 21.6(a) we have redrawn Fig. 21.5 and labeled each of the eight basic feasible solutions with the corresponding value of the objective function as given by (21.11). If somehow we generate x_8 as a basic feasible solution, one can show that the simplex

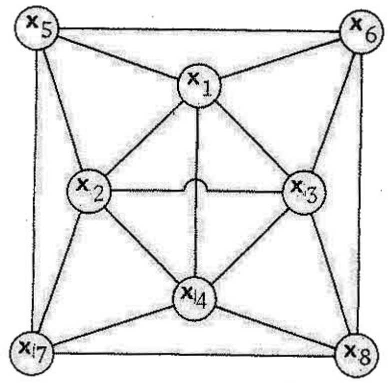

Figure 21.5

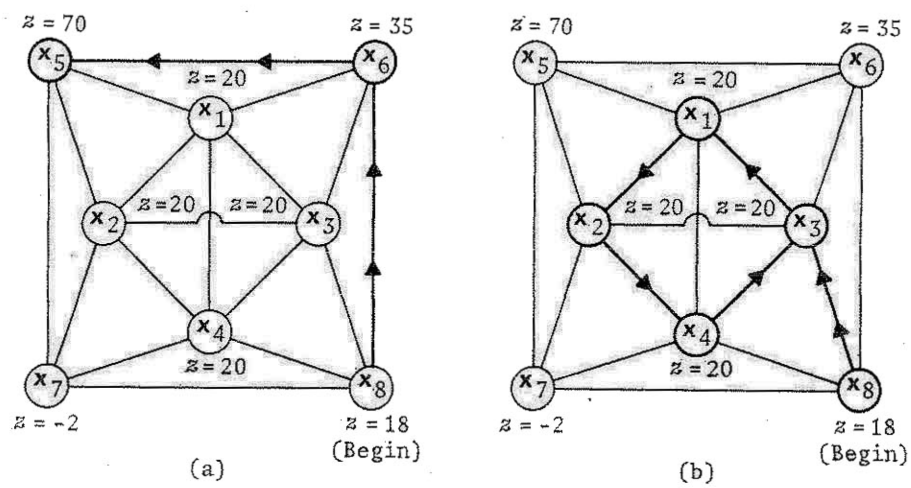

Figure 21.6

method will generate for us the adjacent basic feasible solution x_6. The value of the objective function is thereby increased from $z = 18$ to $z = 35$. From x_6, the simplex method will then generate the adjacent basic feasible solution x_5, where $z = 70$. At all basic feasible solutions adjacent to x_5, the value of the objective function is less than 70. Thus $z = 70$ is the maximum value of the objective function and x_5 is an optimal solution to the problem.

If any of the basic feasible solutions is degenerate, however, it is possible that the simplex method will not lead to an optimal solution. Since the simplex method only guarantees that the value of the objective function will not decrease, it is possible that the situation described in Figure 21.6(b) might arise. Here, we move around the basic feasible solutions x_3, x_1, x_2, and x_4 indefinitely without ever reaching the optimal basic feasible solution x_5. This phenomenon is known as *cycling*. Fortunately, it is not a serious problem in realistic linear programming problems. The circumstances which may produce cycling are rarely encountered in practice; and round-off error in a computer tends to destroy degeneracy so that a loop such as in Fig. 21.6(b) is eventually exited. At any rate, there are algorithms available which can eliminate cycling if it is suspected to be encountered in some specific problem.

EXERCISES

21.1 Convert the following linear programming problem to one in standard form:

$$\text{Minimize} \quad z = 2x_1 + 5x_2$$

subject to

$$\begin{aligned} 3x_1 - 6x_2 &\leq 2 \\ x_1 + x_2 &\leq 3 \\ x_1 \qquad &\geq 6 \\ x_2 &\leq 5 \end{aligned}$$

and

$$x_1, x_2 \geq 0.$$

21.2 Convert the following linear programming problem to one in standard form:

$$\text{Maximize} \quad z = -3x_1 + x_2 + x_3$$

subject to

$$\begin{aligned} 3x_1 - 5x_2 + x_3 &= 3 \\ -2x_1 - x_2 &\leq 2 \\ x_1 &\geq 5 \\ x_2 &\geq 2 \end{aligned}$$

and

$$x_1,\ x_2,\ x_3 \geq 0.$$

21.3 For the following linear programming problem in standard form:

$$\text{Maximize} \quad z = 2x_1 - x_2 + x_3 + 3x_4$$

subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 2 \\ 2x_1 + 4x_3 + x_4 &= 1 \end{aligned}$$

and

$$x_1,\ x_2,\ x_3,\ x_4 \geq 0,$$

(a) show that the feasible set is bounded,
(b) find all basic solutions of the linear system $A\mathbf{x} = \mathbf{b}$,
(c) find all basic feasible solutions of the problem,
(d) find an optimal solution and the maximum value of the objective function,
(e) draw a graph of the basic feasible solutions, as in Fig. 21.5, in which adjacent basic feasible solutions are linked with a single line.

21.4 For the following linear programming problem in standard form:

$$\text{Maximize} \quad z = 2x_1 - 6x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 + 5x_2 + 3x_3 &= 2 \\ -x_1 + 2x_2 + 4x_3 &= 3 \end{aligned}$$

and

$$x_1,\ x_2,\ x_3 \geq 0,$$

(a) find all basic solutions of the linear system $A\mathbf{x} = \mathbf{b}$,
(b) show that the problem has no basic feasible solutions. (From this it follows that the feasible set of this problem is empty.)

21.5 For the following linear programming problem:

$$\text{Maximize} \quad z = 3x_1 + 2x_2 - x_3$$

subject to

$$2x_1 + 3x_2 + x_3 \leq 4$$
$$x_1 + 2x_2 + 3x_3 \leq 5$$

and

$$x_1, x_2, x_3, \geq 0,$$

(a) convert the problem to one in standard form,
(b) find all basic solutions of the linear system $A\mathbf{x} = \mathbf{b}$ in the standard problem,
(c) find all basic feasible solutions of the standard problem,
(d) find an optimal solution of the standard problem and the maximum value of the objective function,
(e) draw a graph of the basic feasible solutions of the standard problem, as in Fig. 21.5, in which adjacent basic feasible solutions are linked with a single line.
(f) find an optimal solution of the original problem.

21.6 Repeat the instructions in Exercise 21.5 for the linear programming problem:

$$\text{Minimize} \quad z = 2x_1 - 3x_2 + x_3$$

subject to

$$x_1 - 2x_2 + 3x_3 \leq 5$$
$$2x_1 + x_2 - 2x_3 = 2$$

and

$$x_1, x_2, x_3 \geq 0.$$

21.7 Prove Theorem 21.1 as follows:

(a) Show that if $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$, then $A\mathbf{x} = \mathbf{b}$ if $\mathbf{x} = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2$ for any t in the interval $[0, 1]$.
(b) Show that if $\mathbf{x}_1 \geq 0$ and $\mathbf{x}_2 \geq 0$, then $\mathbf{x} \geq 0$ if $\mathbf{x} = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2$ for any t in the interval $[0, 1]$.

21.8 Show that the feasible set of the linear programming problem in Example 21.2 is the set illustrated in Fig. 21.3.

21.9 Show that the feasible set of the linear programming problem in Example 21.5 is bounded. Proceed as follows:

(a) From the constraint

$$2x_1 + x_3 + 4x_4 + 2x_5 = 20$$

and the nonnegativity conditions $x_i \geq 0$ ($i = 1,2,3,4,5$) conclude that $x_i \leq 20$ for $i = 1,3,4,5$

(b) Add the two constraints together and conclude that $x_2 \leq 30$.

(c) From (a) and (b), conclude that $\|\mathbf{x}\| \leq r$ for some positive number r.

21.10 Show that the linear system (21.7) can be written in the vector form (21.8).