Ministry of Higher Education
Kingdom of Saudi Arabia

CSTS
SEU, KSA

Linear Algebra (Math 251)
Level IV, Assignment 4
(2016)

1. State whether the following statements are true or false: [6]

   (a) The function $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(x_1, x_2) = (2x_1 + 3x_2, 4x_2 - 1 - x_1, x_1)$ is a linear transformation.

   (a) <u>False</u>

   (b) If $T : V \to W$ be an isomorphism, then $ker(T) = \{0\}$.

   (b) <u>True</u>

   (c) Every square matrix has a LU-decomposition.

   (c) <u>False</u>

   (d) If A is an $m \times n$ matrix, then $A^T A$ is an $m \times m$ matrix.

   (d) <u>False</u>

   (e) In linear programming problems, all variables are restricted to positive values only.

   (e) <u>False</u>

   (f) One of the quickest ways to plot a constraint is to find the two points where the constraint crosses the axes, and draw a straight line between these points.

   (f) <u>True</u>

2. Select one of the alternatives from the following questions as your answer. [6]

   (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator given by $T(x_1, x_2) = (x_2 - x_1, -2x_1 + 2x_2)$. Which of the following vector is in $Ker\ T$?

   A. (-1,2)
   B. (-1,1)
   C. (1,-1)
   D. (1,1)

(b) If $T : M_{44} \to \mathbb{R}^{10}$ be a linear transformation with rank 8, then nullity of T is given by

A. 8

B. 2

C. 4

D. 10

(c) Which of the following sets of eigenvalues have a dominant eigenvalue:

A. $\{6, -4, -6, 1\}$

B. $\{-3, -1, 0, 2\}$

C. $\{-10, 0, 1, 10\}$

D. None of the above

(d) If $B = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$ be a matrix where $B = A^T A$, then the singular values of A are

A. $\{7, 0\}$

B. $\{0, 2\}$

C. $\{7, 2\}$

D. $\{\sqrt{7}, \sqrt{2}\}$

(e) In maximization problem, optimal solution occurring at corner point yields the

A. mean values of z

B. lowest value of z

C. mid values of z

D. highest value of z

(f) Which of the following constraints is not linear?

A. $7A - 6B \leq 45$

B. $X + Y + 3Z \geq 35$

C. $2XY + X = 15$

D. None of the above.

3. Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (-y, x)$ is a linear transformation. [3]

4. Consider the basis $S = \{v_1, v_2\}$ of $\mathbb{R}^2$, where $v_1 = (1, 1),\ v_2 = (0, 1)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the Linear transformation for which $T(v_1) = (2, -1),\ \ T(v_2) = (3, 1)$. Find a formula for $T(x_1, x_2) =?$. [4]

5. Find an LU-decomposition of matrix $A = \begin{bmatrix} 3 & -2 \\ 6 & 4 \end{bmatrix}$. [3]

6. Find the singular values of $A = \begin{bmatrix} 5 & -2 \\ -5 & 2 \end{bmatrix}$. [4]

7. Solve the following LPP by graphical method: [4]

$$\begin{aligned} \max z = x_1 + 2x_2 \quad & \text{subject to}: \\ x_1 + x_2 &> 2 \\ x_2 &< 4 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

3. First property: $T(x + y) = T(x) + T(y)$

Suppose that $x = (x_1, x_2)$ and $y = (y_1, y_2)$, Then,

$T(x + y) = T(x_1 + y_1, x_2 + y_2) = (-(x_2 + y_2), x_1 + y_1) = (-x_2 - y_2, x_1 + y_1)$

And,

$T(x) + T(y) = (-x_2, x_1) + (-y_2 + y_1) = (-x_2 - y_2, x_1 + y_1)$

So,

$T(x + y) = T(x) + T(y)$

Second property: $T(kx) = kT(x)$

Suppose that $x=(x_1,x_2)$

$T(kx) = T(k(x_1,x_2))$

$= T(kx_1 , kx_2) = (-kx_2, kx_1)$

$= k(-x_2, x_1) = kT(x)$

Hence T is a linear transformation.

4. Take any element (x_1, x_2) in R^2, so we can write:

$(x_1, x_2) = c_1v_1 + c_2v_2$

$= c_1 (1, 1) + c_2 (0, 1)$

$= (c_1, c_1) + (0, c_2)$

$= (c_1, c_1 + c_2)$

On comparing both sides, we get

$x_1 = c_1,\ \ x_2 = c_1 + c_2$

$x_2 = x_1 + c_2$ implying $c_2 = x_2 - x_1$

Again,

$(x_1, x_2) = c_1v_1 + c_2v_2$

Since T is linear, so

$T(x_1, x_2) = c_1T(v_1) + c_2T(v_2)$

$= c_1(2, -1) + c_2(4, 3)$

$= (2c_1, -c_1) + (4c_2, 3c_2)$

$= (2c_1 + 4c_2, 3c_2 – c_1)$

$= (2x_1 + 4(x_2 – x_1), 3(x_2 – x_1) –x_1)$

$= (2x_1 + 4x_2 – 4x_1, 3 x_2 –4x_1)$

$T(x_1,x_2)= (4x_2 –2x_1, 3x_2 – 4x_1)$

Which is the required formula.

5. $A = L.U$

$$\begin{bmatrix} 3 & -2 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_1 & 1 \end{bmatrix} . \begin{bmatrix} u_1 & u_2 \\ 0 & u_3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \\ l_1 u_1 & l_1 u_2 + u_3 \end{bmatrix}$$

$u_1 = 3$

$u_2 = -2$

$l_1 u_1 = 6 \implies 3l_1 = 6 \implies l_1 = 2$

$l_1 u_2 + u_3 = 4 \implies 2(-2) + u_3 = 4 \implies u_3 = 8$

So,

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & -2 \\ 0 & 8 \end{bmatrix}$$

Which is the required *LU*-decomposition.

6. $let\ B = A^T A$

$$B = \begin{bmatrix} 5 & -5 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 50 & -20 \\ -20 & 8 \end{bmatrix}$$

The characteristic equation of B is:

$$|B - \lambda I| = 0$$

$$\begin{vmatrix} 50 - \lambda & -20 \\ -20 & 8 - \lambda \end{vmatrix} = 0$$

$$(50 - \lambda)(8 - \lambda) - 400 = 0$$

$$\lambda^2 - 58\lambda = 0$$

$$\lambda(\lambda - 58) = 0$$

The eigenvalues of B are: $\lambda_1 = 0\ and\ \lambda_2 = 58$

Therefore, the singular value of A are:

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{0} = 0$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{58}$$

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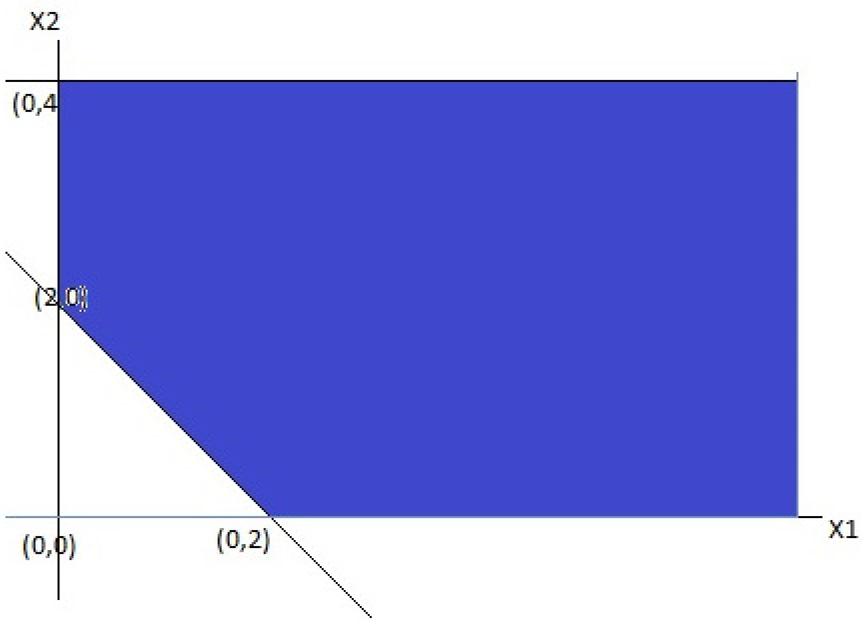
7. For solving it by graphical method, draw the constraints as lines and find the enclosed region

For: $x_1 + x_2 = 2$

x_1	0	2
x_2	2	0

And $x_2=4$

Draw these lines in the first quadrant



Since the feasible region is unbounded so, this problem has no optimal solution.