Ministry of Higher Education
Kingdom of Saudi Arabia

CSTS
SEU, KSA

Linear Algebra (Math 251)
Level IV, Assignment 1
(2016)

1. State whether the following statements are true or false: [6]

   (a) The system of linear equations

$$\begin{aligned} 2x - y &= \frac{1}{2} \\ 12x - 6y &= 3 \end{aligned}$$

   have a unique solution.

   (a) False

   (b) If A is 2×3 and B is 3×4 matrix, then $(AB)^T$ is the matrix of the size 4×2.

   (b) True

   (c) The matrix $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ is not invertible.

   (c) False

   (d) The matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ is lower triangular but not upper triangular.

   (d) False

   (e) The determinant of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 3 & 0 & 3 \end{bmatrix}$ is 3.

   (e) False

   (f) The absolute values of minors and cofactors of the elements of a square matrix are identical.

   (f) True

2. Select one of the alternatives from the following questions as your answer. [6]

(a) If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & -1 \end{bmatrix}$, then $((A^T)^T)^T =$

A. $(A^3)^T$

B. does not exists

C. $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & -1 \end{bmatrix}$

D. $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & -1 \end{bmatrix}$

(b) If $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 10 \\ 4 & 7 \\ -3 & -4 \end{bmatrix}$, then $A + B^T =$

A. addition is not possible

B. $\begin{bmatrix} -2 & 5 & -1 \\ 12 & 8 & -7 \end{bmatrix}$

C. $\begin{bmatrix} -2 & 5 & -1 \\ 12 & 8 & -1 \end{bmatrix}$

D. $\begin{bmatrix} 2 & 5 & -1 \\ 12 & 8 & -1 \end{bmatrix}$

(c) If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, then matrix A is

A. upper triangular.

B. lower triangular.

C. diagonal matrix.

D. all of the above.

(d) The inverse of a lower triangular matrix is

A. upper triangular

B. lower triangular

C. does not exists

D. any matrix

(e) If $B = \begin{bmatrix} 3 & 2 & -1 \\ -1 & 8 & 7 \\ 4 & -3 & 1 \end{bmatrix}$ then the value of minor corresponding to the entry a_{22} is

A. 7

B. -7
C. 1
D. -1

(f) If $A = \begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix}$, then adjoint of A is given by

A. $\begin{bmatrix} -2 & 1 \\ 4 & 2 \end{bmatrix}$

B. $\begin{bmatrix} -2 & 4 \\ 1 & 2 \end{bmatrix}$

C. $\begin{bmatrix} -2 & -1 \\ -4 & 2 \end{bmatrix}$

D. $\begin{bmatrix} 2 & -1 \\ -4 & -2 \end{bmatrix}$

3. Solve the following system of linear equations by Gaussian-elimination method [4]

$$\begin{aligned} x + y + 2z &= 8 \\ 2x + 4y - 3z &= 6 \\ 3x + 6y - 5z &= 8 \end{aligned}$$

Solution: The augmented matrix for the given system is

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 2 & 4 & -3 & 6 \\ 3 & 6 & -5 & 8 \end{bmatrix}$$

Adding -2 times the first row to the second and -3 times the first row to the third gives

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 2 & -7 & -10 \\ 0 & 3 & -11 & -16 \end{bmatrix}$$

Adding -3 times second row to 2 times third row gives

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 2 & -7 & -10 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

The corresponding system of equation is

$$\begin{aligned} x + y + 2z &= 8 \\ 2y - 3z &= -10 \\ -z &= -2 \end{aligned}$$

Successively back substitution gives

$$x = 2, \quad y = 2, \quad z = 2.$$

4. Find the trace of the following matrices: [2]

(a) $\begin{bmatrix} 2 & 1 & -3 \\ 0 & -1 & 4 \\ -2 & -1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}$.

Solution: We know that Trace of a square matrix is sum of main diagonal elements. Therefore

(a) Trace of the matrix=2+(-1)+3=4.

(b) As matrix is not square we can not find trace.

5. Verify the Socks-Shoe property of matrices *i.e.* $(AB)^{-1} = B^{-1}A^{-1}$ [3]
for $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -2 \\ 2 & 0 \end{bmatrix}$.

Solution: If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then we know that $M^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Therefore,

$$\begin{aligned} A^{-1} &= \frac{1}{10}\begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}, \qquad B^{-1} = \frac{1}{4}\begin{bmatrix} 0 & 2 \\ -2 & 3 \end{bmatrix} \\ B^{-1}A^{-1} &= \frac{1}{40}\begin{bmatrix} -6 & 2 \\ -17 & -1 \end{bmatrix} \\ AB &= \begin{bmatrix} -1 & -2 \\ 17 & -6 \end{bmatrix} \\ (AB)^{-1} &= \frac{1}{40}\begin{bmatrix} -6 & 2 \\ -17 & -1 \end{bmatrix} \\ \therefore \quad (AB)^{-1} &= B^{-1}A^{-1} \end{aligned}$$

6. Verify that AA^T is symmetric for $A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \end{bmatrix}$. [3]

Solution: $AA^T = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 26 & 2 \\ 2 & 5 \end{bmatrix}$

Now, $(AA^T)^T = \begin{bmatrix} 26 & 2 \\ 2 & 5 \end{bmatrix} = AA^T$

Therefore , AA^T is symmetric.

7. Find the inverse of the matrix $A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$. [3]

Solution: Here determinant of A, $|A| = -46$.
The matrix of co-factors of the elements of the given matrix is

$$C = \begin{bmatrix} -18 & 2 & 4 \\ -11 & 14 & 5 \\ -10 & -4 & -8 \end{bmatrix}$$

Now, Adjoint of A, $adj(A) = C^T = \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$.

We know that

$$\begin{aligned} A^{-1} &= \frac{1}{|A|} adj(A) \\ &= \frac{1}{-46} \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix} \\ &= \begin{bmatrix} \frac{9}{23} & \frac{11}{46} & \frac{5}{23} \\ -\frac{1}{23} & -\frac{7}{23} & \frac{2}{23} \\ -\frac{2}{23} & -\frac{5}{46} & \frac{4}{23} \end{bmatrix}. \end{aligned}$$

8. Solve the following system of linear equations by Cramer's rule [3]

$$x + y + z = 5$$
$$x - 2y - 3z = -1$$
$$2x + y - z = 3$$

Solution: We know that If $AX = B$ is a system of n linear equations in n unknowns such that $|A| \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, \dots, x_n = \frac{|A_n|}{|A|}$$

where A_j is the matrix obtained by replacing the entries in the j^{th} column of A by the entries in the matrix B.

Therefore, here

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{pmatrix}, \quad So, |A| = 5$$

$$A_1 = \begin{pmatrix} 5 & 1 & 1 \\ -1 & -2 & -3 \\ 3 & 1 & -1 \end{pmatrix}, \quad So, |A_1| = 20$$

$$A_2 = \begin{pmatrix} 1 & 5 & 1 \\ 1 & -1 & -3 \\ 2 & 3 & -1 \end{pmatrix}, \quad So, |A_2| = -10$$

$$A_3 = \begin{pmatrix} 1 & 1 & 5 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \end{pmatrix}, \quad So, |A_3| = 15$$

By Cramer's rule, we have

$$x = \frac{|A_1|}{|A|}, \quad y = \frac{|A_2|}{|A|}, \quad z = \frac{|A_3|}{|A|}$$

Therefore, $x = 4, \quad y = -2, \quad z = 3.$