Ministry of Higher Education
Kingdom of Saudi Arabia

CSTS
SEU, KSA

Linear Algebra (Math 251)
Level IV, Assignment 2
2016-17

1. State whether the following statements are true or false: [6]

   (a) The norm of the vector $u = \frac{1}{\|w\|}.w$ is zero.

   (a) False

   (b) The vectors (3,7) and (3,7,0) are equivalent.

   (b) False

   (c) The set of vectors $\{(2,3,1),\ (-1,1,1),\ (4,6,7)\}$ is linearly independent.

   (c) True

   (d) The set $B = \{(1,2),\ (3,4)\}$ forms a basis of $\mathbb{R}^2$.

   (d) True

   (e) The dimension of a vector space is the number of elements in the largest linearly independent set in that vector space.

   (e) True

   (f) The dimension of row space and column space of a matrix is always same.

   (f) True

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2. Select one of the alternatives from the following questions as your answer. [6]

(a) If $u = (1, 2, 0),\ v = (4, 0, 6)$, then $d(u, v) =$

A. $\sqrt{48}$

B. 7

C. 48

D. 49

(b) If $u = (7, 3, -4, 5)$ and $v = (2, 1, -1, 0)$ then $u.v =$

A. $\sqrt{21}$

B. 13

C. 21

D. 12

(c) The set $A = \{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\}$ forms a basis of the vector space

A. M_{32}

B. M_{22}

C. M_{33}

D. M_{23}

(d) If $v = (2, 1, -2)$ and $\|kv\| = 12$, then the value of k

A. 4

B. $\frac{5}{2}$

C. -$\frac{5}{2}$

D. 3

(e) If $A_{n\times n}$ is a square matrix such that $|A| \neq 0$, then which of the following is/are correct

A. nullity of $A = 0$.

B. rank of $A = n$.

C. A is invertible.

D. all of the above.

(f) If A is $m \times n$ matrix, then

A. rank $(A) = n$

B. rank $(A) = m$

C. rank $(A) \leq \min(m, n)$

D. rank $(A) = m.n$

3. (a) Normalized the vector $u = (1, 2, -2)$. [6]

Solution: Since $\|u\| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$.

and $v = \dfrac{u}{\|u\|} = \dfrac{(1,2,-2)}{3} = \left(\dfrac{1}{3}, \dfrac{2}{3}, -\dfrac{2}{3}\right)$, which is the required normalized vector.

(b) Find the angle between the vectors $u = (1, 2, 3)$, $v = (3, 2, 1)$.

Solution: If θ be the angle between vectors u and v, then

$$\cos\theta = \frac{u.v}{\|u\|\|v\|}$$

Now, $u.v = 1.3 + 2.2 + 3.1 = 10$, $\|u\| = \sqrt{14}$, $\|v\| = \sqrt{14}$
Therefore,

$$\cos\theta = \frac{10}{\sqrt{14}\sqrt{14}} = \frac{2}{7}$$

$\theta = \cos^{-1}\left(\dfrac{2}{7}\right)$, which is the required angle.

(c) Find $(u \times v) \times w$ for $u = (3, 2, -1)$, $v = (0, 2, -3)$, $w = (2, 6, 7)$.

Solution: Since

$$\begin{aligned} u \times v &= \begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ 0 & 2 & -3 \end{vmatrix} \\ &= -4i + 9j + 6k \\ u \times v &= (-4, 9, 6) \\ \therefore (u \times v) \times w &= \begin{vmatrix} i & j & k \\ -4 & 9 & 6 \\ 2 & 6 & 7 \end{vmatrix} \\ &= 27i + 40j - 42k \\ (u \times v) \times w &= (27, 40, -42) \end{aligned}$$

4. Consider the vectors $u = (1, 2, 3)$ and $v = (2, 3, 1)$ in $\mathbb{R}^3$. Write $w = (1, 3, 8)$ as a linear combination of u and v. [3]

Solution: Suppose $w = c_1u + c_2v$, where c_1 and c_2 are scalars whose values are to be determined.

$$\begin{aligned} w &= c_1u + c_2v \\ (1,3,8) &= c_1(1,2,3) + c_2(2,3,1) \\ (1,3,8) &= (c_1 + 2c_2, 2c_1 + 3c_2, 3c_1 + c_2) \end{aligned}$$

$$\Rightarrow \quad c_1 + 2c_2 = 1, \quad 2c_1 + 3c_2 = 3, \quad 3c_1 + c_2 = 8$$

On solving we get
$c_1 = 3, \quad c_2 = -1$
Therefore,

$$w = 3u - v$$

which is the required linear combination.

5. Determine whether or not the set $\{(1,1,1),\ (1,2,3),\ (2,-1,1)\}$ form a basis of $\mathbb{R}^3$. [3]

Solution: The three vectors in $\mathbb{R}^3$ forms a basis if and only if they are linearly independent. Thus form the matrix whose rows are the given vectors and reduce it to echelon form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

The echelon matrix has no zero rows, so the three vectors are linearly independent and so they do form a basis of $\mathbb{R}^3$.

6. Find the rank and basis of the row space of the matrix $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}$. [3]

Solution: Row reduce to echelon form of the given matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 4 & -6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The two nonzero rows (1,2,0,-1) and (0,-2.-3,-1) of the echelon form of matrix A form a basis for row space of A. So, rank $(A) = 2$ and basis $\{(1,2,0,-1),\ (0,-2.-3,-1)\}$.

7. Find the basis and dimension of the solution space W of the following homogeneous system: [3]

$$\begin{aligned} x + 2y + z - 2t &= 0 \\ 2x + 4y + 4z - 3t &= 0 \\ 3x + 6y + 7z - 4t &= 0 \end{aligned}$$

Solution: Reduce the given system to echelon form

$$\begin{aligned} x + 2y + z - 2t &= 0 \\ 2x + 4y + 4z - 3t &= 0 \\ 3x + 6y + 7z - 4t &= 0 \end{aligned}$$

OR

$$\begin{aligned} x + 2y + z - 2t &= 0 \\ 2z + t &= 0 \\ 4z + 2t &= 0 \end{aligned}$$

OR

$$\begin{aligned} x + 2y + z - 2t &= 0 \\ 2z + t &= 0 \end{aligned}$$

The free variables are y and t, so dim (W)=2.
For the basis vectors:

1. Set $y = 1,\ z = 0$ to obtain $x = -2,\ t = 0$, therefore the solution $u_1 = (-2, 1, 0, 0)$.
2. Set $y = 0,\ z = 1$ to obtain $x = -5,\ t = 2$, therefore the solution $u_2 = (-5, 0, 2, -2)$.

Then $\{u_1, u_2\}$ is a basis of W.