Week 6, chapter 4

General Vector Spaces

n - space : $R^n \longrightarrow$ The set of all ordered n-tuple (a sequence of n real number $(x_1, x_2, \ldots x_n)$)

for ex: $n = 1 \rightarrow R^1$ = 1 - space
= set of all real numbers

$n = 2 \rightarrow R^2$ = 2 - space
= set of all orecdred pair of real numbers (x_1, x_2)

$n = 3 \rightarrow R^3$ = 3 space
= set of all ordered triple of real number (x_1, x_2, x_3)

$\vdots$

Vector space: Let V be a non-empty set on which two operations addition and scalar multiplication are defined $(V, +, \cdot)$ is said to vector space if all the following axioms are satisfied.

for all $u, v, w \in V$ and all scalars k and m

Addition:

(1) $u + v \in V$

(2) $u + v = v + u$

(3) $u + (v + w) = (u + v) + w$

(4) $0 + u = u + 0 = u$

(5) $u + (-u) = (-u) + u = 0$

Scalar Multiplication

(6) If k is any scalar and $u \in V$, then $ku \in V$

(7) $k(u + v) = ku + kv$

(8) $(k + m)u = ku + mu$

(9) $k(mu) = (km)u$

(10) $1 \cdot u = u$

Note: (1) A vector space consist of 4 entities
(1) a set of vector (2) a set of scalar
(3) two operations (addition & Multip)
or
(i) V: non empty
(2) c: scalar
(3) vector addition
(4) scalar multiplication

$(V, +, \cdot)$ is a vector space

(2) $V = \{0\}$ zero vector space

(3) To show that a set is not a vector space, you need only find one axiom that is not satisfied.

(4) How to check the V-S
step 1: Identify the set V of objects that will become vectors
step 2: Verify Axiom 1 and Axiom 6 ie addition & scalar multiplication.
step 3: confirm that ~~Axioms~~ rest of the axioms 2, 3, 4, 5, 7, 8, 9, 10 hold.

Q1 Zero is a vector space
(i) Define addition & scalar multiplication

$0 + 0 = 0$

and $k \cdot 0 = 0$

$\therefore$ 0 is a vector space. we can check all the vector space axioms are satisfied

Q2. The set of all Integers is not a vector space

Soln: Let $1 \in V$ — vector space, $\frac{1}{2} \in R$ — Real

Define the addition & scalar multiplication as

(i) $\underset{\text{Integer}}{1} + \underset{\text{Integer}}{2} = 3 \rightarrow$ Integer. It is closed under vector addition

(2) $\frac{1}{2}(1) = \frac{1}{2} \notin V$ It is not closed under scalar multiplication

Q3. The set of all second degree polynomials is not a vector space

Soln: Let $P(x) = x^2$

$q(x) = -x^2 + x + 1$

(i) vector addition

$$P(x) + q(x) = \cancel{x^2} + \cancel{(-x^2)} + x + 1$$

$$= x + 1 \notin V$$

it is not closed under vector addition.

Subspace: A subset W in a vector space V is called a subspace of V, if W is itself a vector space under the addition and scalar multiplication defined in V.

Note: (1) Every vector space V has atleast two subspaces

(1) Zero vector space $\{0\}$ is a subspace of V

(2) V is a subspace of V

Note (2) If W is a non empty subset of a V.S. V, Then W is a subspace of V iff The following conditions holds.

(1) If u and v are in W, Then u + v is in W

(2) If u in W and c is any scalar, Then cu is in W.

Ex: Show that $U = \{(a, b, c) \mid a = b = c\}$ is a subspace in $\mathbb{R}^3$

Soln: clearly $U = \{\phi\}$ as $(0,0,0) \in U$ $(\because 0 = 0 = 0)$

Take any $u, v \in U$ Then

$U = (a, a, a)$ $v = (b, b, b)$

Now

(1) $u + v = (a, a, a) + (b, b, b)$

$= (a+b, a+b, a+b) \in U$

(2) $kU = k(a, a, a)$

$= (ka, ka, ka) \in U$

$\therefore$ U is a V.S of $\mathbb{R}^3$

Ex 2: Show that $W = \{(a,b,c) \mid b = a + c\}$ (1) is a sub space of a v.s in $\mathbb{R}^3$

Soln: Take any $u, v \in W$, then

$u = (u_1, u_2, u_3)$ such that $u_2 = u_1 + u_3$ (2) from (1)

$v = (v_1, v_2, v_3)$ " $v_2 = v_1 + v_3$ (3) from (1)

clearly $W \neq \{\phi\}$ as $(0,0,0) \in W$ as $0 = 0 + 0$ from (1)

(1) $u + v = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3)$

$= (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

from (1) such that $u_2 + v_2 = (u_1 + u_3) + (v_1 + v_3)$ from (2) & (3)

$= (u_1 + v_1) + (u_3 + v_3)$

$\Rightarrow u + v \in W$

(2) $kU = k(u_1, u_2, u_3)$

$= (ku_1, ku_2, ku_3)$

from (1) such that $ku_2 = ku_1 + ku_3$

$\Rightarrow ku \in W$

$\Rightarrow W$ is a sub space of $\mathbb{R}^3$

Ex 3: W is a set of singular matrix of order 2. Show that W is not a subspace of $M_{2\times2}$ with the standard operations.

(singular matrix $|A| = 0$)

Soln: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W$

since $|A| = 0$ & $|B| = 0$ $\therefore$ singular

Now

(1) $A + B = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq 0 \notin W$

is not a sub space

Linear Combination: A vector v in a vector space V is called a linear combination of the vectors $u_1, u_2 \dots u_n$ in V. If v can be written in the form

$$v = c_1 u_1 + c_2 u_2 + \dots c_n u_n \qquad c_1, c_2 \dots c_n \text{ scalars}$$

Q. **Finding a linear combination:**

Let $u = (1, 2, -1)$ and $v = (6, 4, 2) \in \mathbb{R}^3$. Show that $w = (9, 2, 7)$ is a linear combination of u & v.

Soln: Suppose that

$$w = au + bv \quad \text{—— (1)}$$

$$(9, 2, 7) = a(1, 2, -1) + b(6, 4, 2)$$

$$= (a, 2a, -a) + (6b, 4b, 2b)$$

$$= (a + 6b, 2a + 4b, -a + 2b)$$

on equating the corresponding components, we get

$$a + 6b = 9 \quad \text{—— (2)}$$

$$2a + 4b = 2 \quad \text{—— (3)}$$

$$-a + 2b = 7 \quad \text{—— (4)}$$

on adding (2) & (4) we get

$$a - a + 6b + 2b = 9 + 17$$

$$0 + 8b = 16$$

$$b = \frac{16}{8} = 2$$

$$\boxed{b = 2}$$

using this value in (2)

$$a + 6 \times 2 = 9$$

$$a = 9 - 12 = -3$$

$$\boxed{a = -3}$$

Putting the values of a & b in (1)

$$w = -3u + 2v$$

... combination of u & V.

Linear Independent (L.I) and Linear Dependent (L.D)

$S = \{v_1, v_2, \ldots v_k\}$: a set of vectors in a V.S. V

$c_1 v_1 + c_2 v_2 + \ldots c_k v_k = 0$

(i) If all the equation has only the trivial solution $(c_1 = c_2 \ldots c_k = 0)$ then S is call L.I

(2) If not all zero then S is called L.D.

Q1. The standard unit vectors in $\mathbb{R}^3$ in L.I

Soln: In $\mathbb{R}^3$ $\quad i = (1,0,0) \quad j = (0,1,0) \quad k = (0,0,1)$

Let

$c_1 i + c_2 j + c_3 k = 0$ (zero vector)

$c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$

$(c_1, c_2, c_3) = (0,0,0)$

$\Rightarrow c_1 = 0, \quad c_2 = 0, \quad c_3 = 0$

$\Rightarrow S = \{i, j, k\}$ is L.I set

Q2. Determine whether the set $S = \{(1,-2,3)\ (5,6,-1)\ (3,2,1)\}$ is L.I or L.D.

Soln: Let

$c_1(1,-2,3) + c_2(5,6,-1) + c_3(3,2,1) = (0,0,0)$

$(c_1 + 5c_2 + 3c_3,\ -2c_1 + 6c_2 + 2c_3,\ 3c_1 - c_2 + c_3) = (0,0,0)$

on equating corresponding components on both side

$c_1 + 5c_2 + 3c_3 = 0$

$-2c_1 + 6c_2 + 2c_3 = 0$

$3c_1 - c_2 + c_3 = 0$

Now find the value of $c_1, c_2\ c_3$ by yourself

if all $c_1 = c_2 = c_3 = 0$ then L.I if any c is not 0 then L.D.

Note

(1) If a vector zero is in the set, then set is L.D.

(2) Two vectors v_1 and v_2 are L.D iff one of them is a multiple of the others

(3) Every subset of a L.I set is L.I

(4) If you want to check that the set of vectors, say, $\{(1,2,1)\ (2,9,0)\ (3,3,4)\}$ are L.I or L.D, just find the determinant

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 9 & 0 \\ 3 & 3 & 4 \end{vmatrix}$$

If this determinant is non zero, then set is L.I and if the value of the determinant is zero then set is L.D.

Basis and Dimension:

If V is any V.S and $S = \{u_1, u_2, \dots u_n\}$ is a finite set in V. Then S is called a basis for V if

(1) S is L.I

(2) S spans V ie each element of V can be expressed as a linear combination of the elements of S.

Dimension: The dimension of a V.S. V is denoted by $\dim(V)$ and is defined to be the no. of elements in the basis for V.

Note: (1) $\dim(R^3) = 3$, $\dim(R^4) = 4$, $\dim(R^n) = n$

(2) Dimension of the vector space of 2×2 matrix $M_{2\times 2}$ is 4. In general
$$\dim(M_{m\times n}) = m \cdot n$$

(3). If V is a n-dimensional (ie basis has n elements) vector space, then

(1) any set having more that n vectors is L.D

(2) any set having fewer elements than n, then it does not