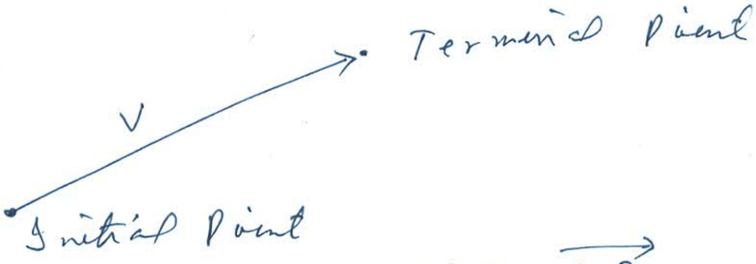
Week-4,5 chapter 3

Vectors

Vectors can be represent in 2-D or 3-D

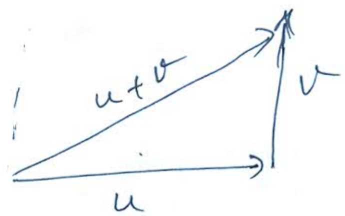


$V = \overrightarrow{AB}$

The zero vector can be denoted by 0, in Two space

$0 = (0,0)$ in 3-space $0 = (0,0,0)$

<u>Addition</u>



<u>Components of vector</u>: If we have 2-vectors $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ Then components of vector are given by

$$\overrightarrow{P_1 P_2} = (x_2 - x_1, y_2 - y_1)$$

Q1 Find the components of a vector with initial point $P_1\ (2,-1,4)$ and terminal point $P_2\ (7,5,-8)$ are

$$\overrightarrow{P_1 P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

$$= (7-2,\ 5-(-1),\ (-8)-4)$$

$$= (5, 6, -12)$$

Q2 If $v = (1,-3,2)$ and $w = (4,2,1)$, find $v+w$ and $v-w$

<u>Sol:</u> $v + w = (5,-1,3)$ and ~~$-w = (-4,-2,-1)$~~

$v - w = (-3,-5,1)$

Theorem 3.1.1 If u, v and w are vectors in R^n, and if k and m are scalars, then

(a) $u + v = v + u$ (b) $(u+v)+w = u+(v+w)$

(c) $u + 0 = 0 + u = u$ (d) $k(u+v) = ku + kv$

(e) $k(mu) = (km)u$ (f) $1 \cdot u = u$

(g) $0 \cdot v = 0$ (h) $k \cdot 0 = 0$

(i) $(-1)v = -v$

Norm of a vector: The Length of a vector v is denoted by $\|v\|$ which is called norm of v or The length of v, or the magnitude of v. so for a vector (v_1, v_2) in R^2

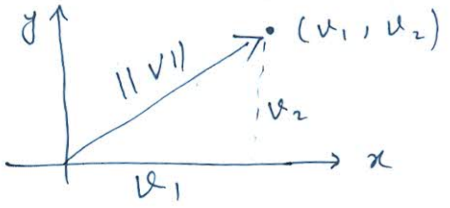
$$\|v\| = \sqrt{v_1^2 + v_2^2}$$

Similarly for a vector (v_1, v_2, v_3) in R^3

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Q1 Find the norm of a vector $v = (-3, 2, 1)$ in R^3.

Soln: $\|v\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$



Unit vector

$$u = \frac{1}{\|v\|} v$$

Standard ~~vec~~ unit vectors in R^2 or R^3

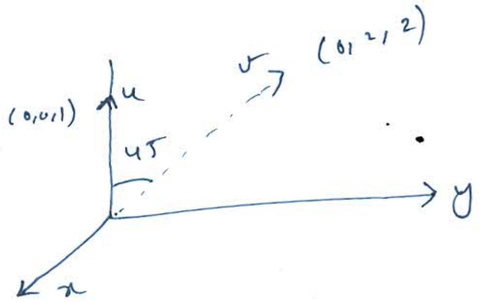
in R^2 $\quad i = (1, 0)$ and $j = (0, 1)$

in R^3 $\quad i = (1, 0, 0)$, $j = (0, 1, 0)$ $k = (0, 0, 1)$

Dot Product: If θ is the angle b/w $\vec{u}$ and $\vec{v}$, Then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

or $$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|v\|}$$



Q Find the dot product of figure

Sol:
$$\|u\| = \sqrt{0^2 + 0^2 + 1^2} = 1$$
$$\|v\| = \sqrt{0^2 + 2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$
$$\cos(45^\circ) = \frac{1}{\sqrt{2}}$$

Now
$$u \cdot v = \|u\| \, \|v\| \cos\theta$$
$$= (1)(2\sqrt{2})(1/\sqrt{2})$$
$$= 2$$

Q Find the dot product of $(u - 2v) \cdot (3u + 4v)$

Sol:
$$(u - \cancel{a}2v) \cdot (3u + 4v) = u \cdot (3u + 4v) - 2v \cdot (3u + 4v)$$
$$= 3(u \cdot u) + 4(u \cdot v) - 6(v \cdot u) - 8(v \cdot v)$$
$$= 3\|u\|^2 - 2(u \cdot v) - 8\|v\|^2$$

Q Calculate the dot product of $a = (1, 2, 3)$ and $b = (4, -5, 6)$

Sol:
$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$
$$= 1(4) + 2(-5) + 3(6) = 4 - 10 + 18 = 12$$

Q3. If $a = (6,-1,3)$ for what value of c is the vector $b = (4, c, -2)$ perpendicular to a?

Sol:
$$a \cdot b = 6(4) - 1(c) + 3(-2).$$
$$= 24 - c - 6$$
$$a \cdot b = 18 - c$$

Now given $a \perp b$ it means $\cos 90 = 0$ so

$$18 - c = 0$$
$$18 = c$$

Orthogonality: Two vectors are orthogonal

iff $u \cdot v = 0$

Q4. Show that $u = (-2, 3, 1, 4)$ and $v = (1, 2, 0, -1)$ are orthogonal in R^4

Sol:
$$u \cdot v = (-2)(1) + (3)(2) + (1)(0) + 4(-1)$$
$$= -2 + 6 - 4$$
$$= 0$$

$\therefore$ vectors are orthogonal

Theorem 3.3.1 If a and b are constants and non-zero then equation of the form $ax + by + c = 0$ represent a line in R^2 with normal n

Theorem 3.3.4: In R^2 The distance b/w The point $P_0(x_0, y_0)$ and the line $ax + by + c = 0$ is $D = \dfrac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$

Similarly in R^3

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Q1 Find the distance b/w the point $(1,-4,-3)$ and the plane $2x - 3y + 6z = -1$

Soln: or we can write $2x - 3y + 6z + 1 = 0$

we know $D = \dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$

$$= \frac{|2(1) + (-3)(-4) + 6(-3) + 1|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{|-3|}{7}$$

$$= \frac{3}{7}$$

Q2. The two planes are given $x + 2y - 2z = 3$ and $2x + 4y - 4z = 7$ are $\parallel$. Find the distance b/w them.

Soln: we are given $x + 2y - 2z - 3 = 0$ — (1)

and $2x + 4y - 4z - 7 = 0$ — (2)

Let us choose any point on (1) by putting $x = y = 0$ Then we get $x = 3$ So ~~this~~ Th point on the plane (1) is $(3,0,0)$

Now we ~~k~~ know the formula of Dist.

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|2(3) + 4(0) + (-4)(0) - 7|}{\sqrt{2^2 + 4^2 + (-4)^2}} = \frac{1}{6}$$

Defn: If x_0 and $\vec{v}$ are vectors in R^n, and if v is non-zero, Then the equation $x = x_0 + tv$ defines a line through x_0 that is $\parallel$ to v. If $x_0 = 0$ Then the line pass through the origin.

Similarly $x = x_0 + t_1v_1 + t_2v_2$ for R^3

Q1 Find a vector equation and parametric equation of line in R^3 that pass through the point $P_0(1,2,-3)$ and is parallel to the vector $v = (4,-5,1)$

Soln: we know the vector equation of the line is $\vec{x} = x_0 + tv$

here $x_0 = (1,2,-3)$ and $\vec{v} = (4,-5,1)$ so

$$(x,y,z) = (1,2,-3) + t(4,-5,1)$$

Equating the corresponding components on the two sides of this equation

$$x = 1+4t, \quad y = 4-5t, \quad z = -3+t.$$

3.5 Cross Product : If $\vec{u} = (u_1,u_2,u_3)$ and $\vec{v} = (v_1,v_2,v_3)$

Then The cross product $\vec{u}\times\vec{v} = (u_2v_3 - u_3v_2, \; u_3v_1 - u_1v_3, \; u_1v_2 - u_2v_1)$

or we can write

$$\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$ Then $u\times v = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$

Q1 calculate a cross product when $\vec{u} = (1,2,-2)$, $\vec{v} = (3,0,1)$

Soln: we can write

$$\begin{bmatrix} 1 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix}$$ Then $u\times v = \left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \right)$

$$= (2, -7, -6)$$

Theorem: If $\vec{u}, \vec{v}, \vec{w}$ are vectors in 3 space, Then

(a) $u\cdot(u\times v) = 0$

(b) $v\cdot(u\times v) = 0$

(c) $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$

(d) $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v}) + (u\times w)$

(e) $u\times u = 0$

(f) $u\times 0 = 0$

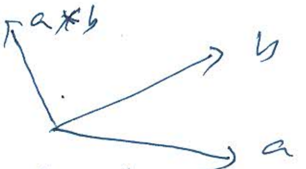
Standard unit vector

$i\times i = 0$ $\quad j\times j = 0$ $\quad k\times k = 0$

$i\times j = k$ $\quad j\times k = i$ $\quad k\times i = j$

$= -k$ $\quad k\times j = -i$ $\quad i\times j = -j$

Defn of cross Product : The cross product of $\vec{a} \times \vec{b}$ of two vectors is another vector that is at right angles to both.



right hand rule

Q. If $\vec{a} = (3,-3,1)$ ~~(2,-1,-1)~~ and ~~b = (-~~

a $b = (4,9,2)$ ~~(-3,4,1)~~ find (a) $\vec{a} \times \vec{b}$ ~~(b) $\vec{b} \times \vec{a}$~~

Soln: we kn $\cancel{\begin{bmatrix} 2 & -1 & -1 \\ -3 & 4 & 1 \end{bmatrix} \Rightarrow a \times b = \left(\begin{vmatrix} -1 & -1 \\ 4 & 1 \end{vmatrix} - \begin{vmatrix} 2 & -1 \\ -3 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -3 & 4 \end{vmatrix} \right.}$

$=$

$$\begin{bmatrix} 3 & -3 & 1 \\ 4 & 9 & 2 \end{bmatrix} \Rightarrow a \times b = \left(\begin{vmatrix} -3 & 1 \\ 9 & 2 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 3 & -3 \\ 4 & 9 \end{vmatrix} \right)$$

$$= (-15, -2, 39)$$

or $= -15i - 2j + 39k$ //

Q Find the cosine of angle θ between the vectors $u = (6,3,2)$ and $v = (2,3,6)$

Soln:
$$\cos\theta = \frac{u \cdot v}{\|u\| \, \|v\|}$$

$\therefore \|v\| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$

$\|u\| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$

$(6,3,2) \cdot (2,3,6) = 12 + 9 + 12 = 33$

$$= \frac{33}{49}$$

Q. Calculate $u \cdot (v \times w)$ where $u = (1,-3,4)$ and $v = (2i - 3j + k)$ $w = -i + 4j - 3k$

Soln:
$$v \times w = \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ -1 & 4 & -3 \end{vmatrix} \Rightarrow i \begin{vmatrix} -3 & 1 \\ 4 & -3 \end{vmatrix} - j \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} + k \begin{vmatrix} 2 & -3 \\ -1 & 4 \end{vmatrix}$$

$\Rightarrow 5i + 5j + 5k$

Now $u \cdot (v \times w) = (1,-3,4) \cdot (5,5,5)$

$= 5 - 15 + 20$

$= 10$

Q Show that $u(2,1,0)$ and $v(-3,6,5)$ are orthogonal

Soln: $u \cdot v = 2 \times (-3) + (1)(6) + 0 \times 5$

$= 0$

So u & v are orthogonal vectors.