Chapter 4

Vector Spaces

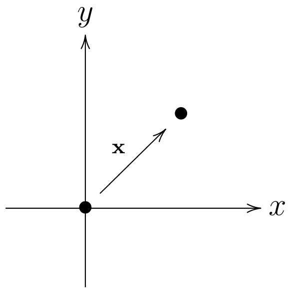
4.1 Vectors in $\mathbb{R}^n$

Homework: [Textbook, §4.1 Ex. 15, 21, 23, 27, 31, 33(d), 45, 47, 49, 55, 57; p. 189-].

We discuss vectors in plane, in this section.

In physics and engineering, a vector is represented as a directed segment. It is determined by a length and a direction. We give a short review of vectors in the plane.

Definition 4.1.1 A **vector x in the plane** is represented geometrically by a *directed line segment* whose *initial point* is the origin and whose terminal point is a point (x_1, x_2) as shown in in the textbook, page 180.



The bullet at the end of the arrow is the terminal point (x_1, x_2). (*See the textbook,page 180 for a better diagram.*) This vector is represented by the same *ordered pair* and we write

$$\mathbf{x} = (x_1, x_2).$$

1. We do this because other information is superfluous. Two vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are equal if $u_1 = v_1$ and $u_2 = v_2$.

2. Given two vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, we define **vector addition**
$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2).$$
See the diagram in the textbook, page 180 for geometric interpretation of vector addition.

3. For a scalar c and a vector $\mathbf{v} = (v_1, v_2)$ define
$$c\mathbf{v} = (cv_1, cv_2)$$
See the diagram in the textbook, page 181 for geometric interpretation of scalar multiplication.

4. Denote $-\mathbf{v} = (-1)\mathbf{v}$.

Reading assignment: Read [Textbook, Example 1-3, p. 180-] and study all the diagrams.

Obvioulsly, these vectors behave like row matrices. Following list of properties of vectors play a fundamental role in linear algebra. In fact, in the next section these properties will be abstracted to define vector spaces.

Theorem 4.1.2 Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three vectors in the plane and let c, d be two scalar.

1. $\mathbf{u}+\mathbf{v}$ is a vector in the plane	*closure under addition*
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$	*Commutative property of addition*
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$	*Associate property of addition*
4. $(\mathbf{u}+\mathbf{0})=\mathbf{u}$	*Additive identity*
5. $\mathbf{u}+(-1)\mathbf{u}=\mathbf{0}$	*Additive inverse*
6. $c\mathbf{u}$ is a vector in the plane	*closure under scalar multiplication*
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$	*Distributive propertyof scalar mult.*
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$	*Distributive property of scalar mult.*
9. $c(d\mathbf{u})=(cd)\mathbf{u}$	*Associate property of scalar mult.*
10. $1(\mathbf{u})=\mathbf{u}$	*Multiplicative identity property*

Proof. Easy, see the textbook, papge 182.

4.1.1 Vectors in $\mathbb{R}^n$

The discussion of vectors in plane can now be extended to a discussion of vectors in $n-$space. A vector in $n-$space is represented by an **ordered** $n-$**tuple** $(x_1, x_2, \ldots, x_n)$.

The set of all ordered $n-$tuples is called the $n-$space and is denoted by $\mathbb{R}^n$. So,

1. $\mathbb{R}^1 = 1 - space =$ set of all real numbers,

2. $\mathbb{R}^2 = 2 - space$ = set of all ordered pairs (x_1, x_2) of real numbers
3. $\mathbb{R}^3 = 3 - space$ = set of all ordered triples (x_1, x_2, x_3) of real numbers
4. $\mathbb{R}^4 = 4 - space$ = set of all ordered quadruples (x_1, x_2, x_3, x_4) of real numbers. (*Think of space-time.*)
5. ......
6. $\mathbb{R}^n = n{-}space$ = set of all ordered ordered $n-$tuples $(x_1, x_2, \ldots, x_n)$ of real numbers.

Remark. We do not distinguish between points in the $n-$space $\mathbb{R}^n$ and **vectors** in $n-$space (defined similalry as in definition 4.1.1). This is because both are described by same data or information. A vector in the $n-$space $\mathbb{R}^n$ is denoted by (and determined) by an $n-$tuples $(x_1, x_2, \ldots, x_n)$ of real numbers and same for a point in $n-$space $\mathbb{R}^n$. The $i^{th}-$entry x_i is called the $i^{th}-$coordinate.

Also, a point in $n-$space $\mathbb{R}^n$ can be thought of as row matrix. (*Some how, the textbook avoids saying this.*) So, the addition and scalar multiplications can be defined is a similar way, as follows.

Definition 4.1.3 Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be vectors in $\mathbb{R}^n$. The the sum of these two vectors is defined as the vector

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

For a scalar c, define scalar multiplications, as the vector

$$c\mathbf{u} = (cu_1, cu_2, \ldots, cu_n).$$

Also, we define negative of $\mathbf{u}$ as the vector

$$-\mathbf{u} = (-1)(u_1, u_2, \ldots, u_n) = (-u_1, -u_2, \ldots, -u_n)$$

and the difference

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = (u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n).$$

Theorem 4.1.4 All the properties of theorem 4.1.2 hold, for any three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $n-$space $\mathbb{R}^n$ and salars c, d.

Theorem 4.1.5 Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ and let c be a scalar. Then,

1. $\mathbf{v} + \mathbf{0} = \mathbf{v}$.

   (*Because of this property,* $\mathbf{0}$ *is called the* **additive identity** *in* $\mathbb{R}^n$.)

   Further, the additive identitiy unique. That means, if $\mathbf{v} + \mathbf{u} = \mathbf{v}$ for all vectors $\mathbf{v}$ in $\mathbb{R}^n$ than $\mathbf{u} = \mathbf{0}$.

2. Also $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

   (*Because of this property,* $-\mathbf{v}$ *is called the additive inverse of* $\mathbf{v}$.)

   Further, the additive inverse of $\mathbf{v}$ is unique. This means that $\mathbf{v} + \mathbf{u} = \mathbf{0}$ for some vector $\mathbf{u}$ in $\mathbb{R}^n$, then $\mathbf{u} = -\mathbf{v}$.

3. $0\mathbf{v} = \mathbf{0}$.

   Here the 0 on left side is the scalar zero and the bold $\mathbf{0}$ is the vector zero in $\mathbb{R}^n$.

4. $c\mathbf{0} = \mathbf{0}$.

5. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.

6. $-(-\mathbf{v}) = \mathbf{v}$.

Proof. To prove that additive identity is unique, suppose $\mathbf{v} + \mathbf{u} = \mathbf{v}$ for all $\mathbf{v}$ in $\mathbb{R}^n$. Then, taking $\mathbf{v} = \mathbf{0}$, we have $\mathbf{0} + \mathbf{u} = \mathbf{0}$. Therefore, $\mathbf{u} = \mathbf{0}$.

To prove that additive inverse is unique, suppose $\mathbf{v}+\mathbf{u} = \mathbf{0}$ for some vector $\mathbf{u}$. Add $-\mathbf{v}$ on both sides, from left side. So,

$$-\mathbf{v} + (\mathbf{v} + \mathbf{u}) = -\mathbf{v} + \mathbf{0}$$

So,

$$(-\mathbf{v} + \mathbf{v}) + \mathbf{u} = -\mathbf{v}$$

So,

$$\mathbf{0} + \mathbf{u} = -\mathbf{v} \qquad So, \qquad \mathbf{u} = -\mathbf{v}.$$

We will also prove (5). So suppose $c\mathbf{v} = \mathbf{0}$. If $c = 0$, then there is nothing to prove. So, we assume that $c \neq 0$. Multiply the equation by c^{-1}, we have $c^{-1}(c\mathbf{v}) = c^{-1}\mathbf{0}$. Therefore, by associativity, we have $(c^{-1}c)\mathbf{v} = \mathbf{0}$. Therefore $1\mathbf{v} = \mathbf{0}$ and so $\mathbf{v} = \mathbf{0}$.

The other statements are easy to see. The proof is complete. ■

Remark. We denote a vector $\mathbf{u}$ in $\mathbb{R}^n$ by a row $\mathbf{u} = (u_1, u_2, \ldots, u_n)$. As I said before, it can be thought of a row matrix

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \ldots & u_n \end{bmatrix}.$$

In some other situation, it may even be convenient to denote it by a column matrix:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix}.$$

Obviosly, we cannot mix the two (in fact, three) different ways.

Reading assignment: Read [Textbook, Example 6, p. 187].

Exercise 4.1.6 (Ex. 46, p. 189) Let $\mathbf{u} = (0, 0, -8, 1)$ and $\mathbf{v} = (1, -8, 0, 7)$. Find $\mathbf{w}$ such that $2\mathbf{u} + \mathbf{v} - 3\mathbf{w} = \mathbf{0}$.

Solution: We have

$$\mathbf{w} = \frac{2}{3}\mathbf{u} + \frac{1}{3}\mathbf{v} = \frac{2}{3}(0, 0, -8, 1) + \frac{1}{3}(1, -8, 0, 7) = (\frac{1}{3}, -\frac{8}{3}, -\frac{16}{3}, 3).$$

Exercise 4.1.7 (Ex. 50, p. 189) Let $\mathbf{u_1} = (1, 3, 2, 1)$, $\mathbf{u_2} = (2, -2, -5, 4)$, $\mathbf{u_3} = (2, -1, 3, 6)$. If $\mathbf{v} = (2, 5, -4, 0)$, write $\mathbf{v}$ as a linear combination of $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$. If it is not possible say so.

Solution: Let $\mathbf{v} = a\mathbf{u_1} + b\mathbf{u_2} + c\mathbf{u_3}$. We need to solve for a, b, c. Writing the equation explicitly, we have

$$(2, 5, -4, 0) = a(1, 3, 2, 1) + b(2, -2, -5, 4) + c(2, -1, 3, 6).$$

Therefore

$$(2, 5, -4, 0) = (a + 2b + 2c, 3a - 2b - c, 2a - 5b + 3c, a + 4b + 6c)$$

Equating entry-wise, we have system of linear equation

$$\begin{array}{rrrl} a & +2b & +2c & = 2 \\ 3a & -2b & -c & = 5 \\ 2a & -5b & +3c & = -4 \\ a & +4b & +6c & = 0 \end{array}$$

We write the augmented matrix:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{bmatrix}$$

We use TI, to reduce this matrix to Gauss-Jordan form:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, the system is consistent and $a = 2, b = 1, c = -1$. Therefore

$$\mathbf{v} = 2\mathbf{u_1} + \mathbf{u_2} - \mathbf{u_3},$$

which can be checked directly,

4.2 Vector spaces

Homework: [Textbook, §4.2 Ex.3, 9, 15, 19, 21, 23, 25, 27, 35; p.197].

The main point *in the section is to define vector spaces and talk about examples.*

The following definition is an **abstruction** of theorems 4.1.2 and theorem 4.1.4.

Definition 4.2.1 Let V be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and scalars c and d, then V is called a **vector space** (over the reals $\mathbb{R}$).

1. Addition:
   (a) $\mathbf{u} + \mathbf{v}$ is a vector in V (*closure under addition*).
   (b) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (*Commutative property of addition*).
   (c) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ (*Associative property of addition*).
   (d) There is a **zero vector** $\mathbf{0}$ in V such that for every $\mathbf{u}$ in V we have $(\mathbf{u} + \mathbf{0}) = \mathbf{u}$ (*Additive identity*).
   (e) For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ (*Additive inverse*).
2. Scalar multiplication:
   (a) $c\mathbf{u}$ is in V (*closure under scalar multiplication*0.

(b) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (*Distributive propertyof scalar mult.*).

(c) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (*Distributive property of scalar mult.*).

(d) $c(d\mathbf{u}) = (cd)\mathbf{u}$ (*Associate property of scalar mult.*).

(e) $1(\mathbf{u}) = \mathbf{u}$ (*Scalar identity property*).

Remark. It is important to realize that a vector space consisits of four entities:

1. A set V of vectors.
2. A set of scalars. In this class, it will alawys be the set of real numbers $\mathbb{R}$. (Later on, this could be the set of complex numbers $\mathbb{C}$.)
3. A vector addition denoted by $+$.
4. A scalar multiplication.

Lemma 4.2.2 We use the notations as in definition 4.2.1. First, the zero vector $\mathbf{0}$ is unique, satisfying the property (1d) of definition 4.2.1.

Further, for any $\mathbf{u}$ in V, the **additive inverse** $-\mathbf{u}$ is unique.

Proof. Suppose, there is another element θ that satisfy the property (1d). Since $\mathbf{0}$ satisfy (1d), we have

$$\theta = \theta + \mathbf{0} = \mathbf{0} + \theta = \mathbf{0}.$$

The last equality follows because θ satisfes the property(1d).

(*The proof that additive inverse of* $\mathbf{u}$ *unique is similar the proof of theorem 2.3.2, regarding matrices.*) Suppose $\mathbf{v}$ is another additive inverse of $\mathbf{u}$.

$$\mathbf{u} + \mathbf{v} = \mathbf{0} \qquad \textit{and} \qquad \mathbf{u} + (-\mathbf{u}) = \mathbf{0}.$$

So.

$$-\mathbf{u} = \mathbf{0} + (-\mathbf{u}) = (\mathbf{u} + \mathbf{v}) + (-\mathbf{u}) = \mathbf{v} + (\mathbf{u} + (-\mathbf{u})) = \mathbf{v} + \mathbf{0} = \mathbf{v}.$$

So, the proof is complete. ■

Reading assignment: Read [Textbook, Example 1-5, p. 192-]. These examples lead to the following list of important examples of vector spaces:

Example 4.2.3 Here is a collection examples of vector spaces:

1. The set $\mathbb{R}$ of real numbers $\mathbb{R}$ is a vector space over $\mathbb{R}$.
2. The set $\mathbb{R}^2$ of all ordered pairs of real numers is a vector space over $\mathbb{R}$.
3. The set $\mathbb{R}^n$ of all ordered $n-$tuples of real numersis a vector space over $\mathbb{R}$.
4. The set $C(\mathbb{R})$ of all continuous functions defined on the real number line, is a vector space over $\mathbb{R}$.
5. The set $C([a, b]))$ of all continuous functions defined on interval $[a, b]$ is a vector space over $\mathbb{R}$.
6. The set $\mathbb{P}$ of all polynomials, with real coefficients is a vector space over $\mathbb{R}$.
7. The set $\mathbb{P}_n$ of all polynomials of degree $\leq n$, with real coefficients is a vector space over $\mathbb{R}$.
8. The set $\mathbb{M}_{m,n}$ of all $m \times n$ matrices, with real entries, is a vector space over $\mathbb{R}$.

Reading assignment: Read [Textbook, Examples 6-6].

Theorem 4.2.4 Let V be vector space over the reals $\mathbb{R}$ and $\mathbf{v}$ be an element in V. Also let c be a scalar. Then,

1. $0\mathbf{v} = \mathbf{0}$.
2. $c\mathbf{0} = \mathbf{0}$.
3. If $c\mathbf{v} = \mathbf{0}$, then either $c = 0$ or $\mathbf{v} = \mathbf{0}$.
4. $(-1)\mathbf{v} = -\mathbf{v}$.

Proof. We have to prove this theorem using the definition 4.2.1. Other than that, the proof will be similar to theorem 4.1.5. To prove (1), write $\mathbf{w} = 0\mathbf{v}$. We have

$$\mathbf{w} = 0\mathbf{v} = (0+0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v} = \mathbf{w} + \mathbf{w} \quad (by\ distributivity Prop.(2c)).$$

Add $-\mathbf{w}$ to both sides

$$\mathbf{w} + (-\mathbf{w}) = (\mathbf{w} + \mathbf{w}) + (-\mathbf{w})$$

By (1e) of 4.2.1, we have

$$\mathbf{0} = \mathbf{w} + (\mathbf{w} + (-\mathbf{w})) = \mathbf{w} + \mathbf{0} = \mathbf{w}.$$

So, (1) is proved. The proof of (2) will be exactly similar.

To prove (3), suppose $c\mathbf{v} = \mathbf{0}$. If $c = 0$, then there is nothing to prove. So, we assume that $c \neq 0$. Multiply the equation by c^{-1}, we have $c^{-1}(c\mathbf{v}) = c^{-1}\mathbf{0}$. Therefore, by associativity, we have $(c^{-1}c)\mathbf{v} = \mathbf{0}$. Therefore $1\mathbf{v} = \mathbf{0}$ and so $\mathbf{v} = \mathbf{0}$.

To prove (4), we have

$$\mathbf{v} + (-1)\mathbf{v} = 1.\mathbf{v} + (-1)\mathbf{v} = (1-1)\mathbf{v} = 0.\mathbf{v} = \mathbf{0}.$$

This completes the proof. ■

Exercise 4.2.5 (Ex. 16, p. 197) Let V be the set of all fifth-degree polynomials with standared operations. Is it a vector space. Justify your answer.

Solution: In fact, V is not a vector space. Because V is not closed under addition(axiom (1a) of definition 4.2.1 fails): $f = x^5 + x - 1$ and $g = -x^5$ are in V but $f + g = (x^5 + x - 1) - x^5 = x - 1$ is not in V.

Exercise 4.2.6 (Ex. 20, p. 197) Let $V = \{(x, y) : x \geq 0, y \geq 0\}$ with standared operations. Is it a vector space. Justify your answer.

Solution: In fact, V is not a vector space. Not every element in V has an addditive inverse (axiom i(1e) of 4.2.1 fails): $-(1, 1) = (-1, -1)$ is not in V.

Exercise 4.2.7 (Ex. 22, p. 197) Let $V = \{\left(x, \frac{1}{2}x\right) : x\, real\ number\}$ with standared operations. Is it a vector space. Justify your answer.

Solution: Yes, V is a vector space. We check all the properties in 4.2.1, one by one:

1. Addition:

   (a) For real numbers x, y, We have
   $$\left(x, \frac{1}{2}x\right) + \left(y, \frac{1}{2}y\right) = \left(x + y, \frac{1}{2}(x + y)\right).$$
   So, V is closed under addition.

   (b) Clearly, addition is closed under addition.

   (c) Clearly, addition is associative.

   (d) The element $\mathbf{0} = (0, 0)$ satisfies the property of the zero element.

(e) We have $-\left(x, \frac{1}{2}x\right) = \left(-x, \frac{1}{2}(-x)\right)$ .So, every element in V has an additive inverse.

2. Scalar multiplication:

(a) For a scalar c, we have

$$c\left(x, \frac{1}{2}x\right) = \left(cx, \frac{1}{2}cx\right).$$

So, V is closed under scalar multiplication.

(b) The distributivity $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ works for $\mathbf{u}, \mathbf{v}$ in V.

(c) The distributivity $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ works, for $\mathbf{u}$ in V and scalars c, d.

(d) The associativity $c(d\mathbf{u}) = (cd)\mathbf{u}$ works.

(e) Also $1\mathbf{u} = \mathbf{u}$.

4.3 Subspaces of Vector spaces

We will skip this section, after we just mention the following.

Definition 4.3.1 A nonempty subset W of a vector space V is called a subspace of V if W is a vector space under the operations addition and scalar multiplication defined in V.

Example 4.3.2 Here are some obvious examples:

1. Let $W = \{(x, 0) : x \textit{ is real number}\}$. Then $W \subseteq \mathbb{R}^2$. (*The notation* $\subseteq$ *reads as 'subset of'.*) It is easy to check that W is a subspace of $\mathbb{R}^2$.

2. Let W be the set of all points on any given line $y = mx$ through the origin in the plane $\mathbb{R}^2$. Then, W is a subspace of $\mathbb{R}^2$.

3. Let P_2, P_3, P_n be vector space of polynomials, respectively, of degree less or equal to $2, 3, n$. (See example 4.2.3.) Then P_2 is a subspace of P_3 and P_n is a subspace of P_{n+1}.

Theorem 4.3.3 Suppose V is a vector space over $\mathbb{R}$ and $W \subseteq V$ is a **nonempty** subset of V. Then W is a subspace of V if and only if the following two closure conditions hold:

1. If $\mathbf{u}, \mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W.

2. If $\mathbf{u}$ is in W and c is a scalar, then $c\mathbf{u}$ is in W.

Reading assignment: Read [Textbook, Examples 1-5].

4.4 Spanning sets and linear indipendence

Homework. [Textbook, §4.4, Ex. 27, 29, 31; p. 219].

The main point here is to write a vector as linear combination of a give set of vectors.

Definition 4.4.1 A vector $\mathbf{v}$ in a vector space V is called a **linear combination** of vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_k\mathbf{u_k},$$

where $c_1, c_2, \ldots, c_k$ are scalars.

Definition 4.4.2 Let V be a vector space over $\mathbb{R}$ and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of V. We say that S is a **spanning set** of V if every vector $\mathbf{v}$ of V can be written as a liner combination of vectors in S. In such cases, we say that S **spans** V.

Definition 4.4.3 Let V be a vector space over $\mathbb{R}$ and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of V. Then the **span of** S is the set of all linear combinations of vectors in S,

$$span(S) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} : c_1, c_2, \ldots, c_k \textit{ are scalars}\}.$$

1. The span of S is denoted by $span(S)$ as above or $span\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$.

2. If $V = span(S)$, then say V is spanned by S or S spans V.

Theorem 4.4.4 Let V be a vector space over $\mathbb{R}$ and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of V. Then *span*(S) is a subspace of V.

Further, *span*(S) is the smallest subspace of V that contains S. This means, if W is a subspace of V and W contains S, then *span*(S) is contained in W.

Proof. By theorem 4.3.3, to prove that *span*(S) is a subspace of V, we only need to show that *span*(S) is closed under addition and scalar multiplication. So, let $\mathbf{u}, \mathbf{v}$ be two elements in *span*(S). We can write

$$\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} \quad \textit{and} \quad \mathbf{v} = d_1\mathbf{v_1} + d_2\mathbf{v_2} + \cdots + d_k\mathbf{v_k}$$

where $c_1, c_2, \ldots, c_k, d_1, d_2, \ldots, d_k$ are scalars. It follows

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v_1} + (c_2 + d_2)\mathbf{v_2} + \cdots + (c_k + d_k)\mathbf{v_k}$$

and for a scalar c, we have

$$c\mathbf{u} = (cc_1)\mathbf{v_1} + (cc_2)\mathbf{v_2} + \cdots + (cc_k)\mathbf{v_k}.$$

So, both $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ are in *span*(S), because the are linear combination of elements in S. So, *span*(S) is closed under addition and scalar multiplication, hence a subspace of V.

To prove that *span*(S) is smallest, in the sense stated above, let W be subspace of V that contains S. We want to show *span*(S) is contained in W. Let $\mathbf{u}$ be an element in *span*(S). Then,

$$\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k}$$

for some scalars c_i. Since $S \subseteq W$, we have $v_i \in W$. Since W is closed under addition and scalar multiplication, $\mathbf{u}$ is in W. So, *span*(S) is contained in W. The proof is complete. ■

Reading assignment: Read [Textbook, Examples 1-6, p. 207-].

4.4.1 Linear dependence and independence

Definition 4.4.5 Let V be a vector space. A set of elements (vectors) $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_k}\}$ is said to be **linearly independent** if the equation

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \mathbf{0}$$

has only trivial solution

$$c_1 = 0, c_2 = 0, \ldots, c_k = 0.$$

We say S is **linearly dependent**, if S in not linearly independent. (*This means, that S is said to be linearly dependent, if there is at least one nontrivial (i.e. nonzero) solutions to the above equation.*)

Testing for linear independence

Suppose V is a subspace of the $n-$space $\mathbb{R}^n$. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_k}\}$ be a set of elements (i.e. vectors) in V. To test whether S is linearly independent or not, we do the following:

1. From the equation

   $$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \mathbf{0},$$

   write a homogeneous system of equations in variabled $c_1, c_2, \ldots, c_k$.

2. Use Gaussian elemination (with the help of TI) to determine whether the system has a unique solutions.

3. If the system has only the trivial solution

   $$c_1 = 0, c_2 = 0, \cdots, c_k = 0,$$

   then S is linearly independent. Otherwise, S is linearly dependent.

Reading assignment: Read [Textbook, Eamples 9-12, p. 214-216].

Exercise 4.4.6 (Ex. 28. P. 219) Let $S = \{(6,2,1),(-1,3,2)\}$. Determine, if S is linearly independent or dependent?

Solution: Let

$$c(6,2,1) + d(-1,3,2) = (0,0,0).$$

If this equation has only trivial solutions, then it is linealry independent. This equaton gives the following system of linear equations:

$$\begin{array}{rrl} 6c & -d & = 0 \\ 2c & +3d & = 0 \\ c & +2d & = 0 \end{array}$$

The augmented matrix for this system is

$$\begin{bmatrix} 6 & -1 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix}. \quad its\ gauss - Jordan\ form: \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $c = 0, d = 0$. The system has only trivial (i.e. zero) solution. We conclude that S is linearly independent.

Exercise 4.4.7 (Ex. 30. P. 219) Let

$$S = \left\{ \left(\frac{3}{4}, \frac{5}{2}, \frac{3}{2}\right), \left(3, 4, \frac{7}{2}\right), \left(-\frac{3}{2}, 6, 2\right) \right\}.$$

Determine, if S is linearly independent or dependent?

Solution: Let

$$a\left(\frac{3}{4}, \frac{5}{2}, \frac{3}{2}\right) + b\left(3, 4, \frac{7}{2}\right) + c\left(-\frac{3}{2}, 6, 2\right) = (0,0,0).$$

If this equation has only trivial solutions, then it is linealry independent. This equaton gives the following system of linear equations:

$$\begin{array}{rrrl} \frac{3}{4}a & +3b & -\frac{3}{2}c & = 0 \\ \frac{5}{2}a & +4b & +6c & = 0 \\ \frac{3}{2}a & +\frac{7}{2}b & +2c & = 0 \end{array}$$

The augmented matrix for this system is

$$\begin{bmatrix} \frac{3}{4} & 3 & -\frac{3}{2} & 0 \\ \frac{5}{2} & 4 & 6 & 0 \\ \frac{3}{2} & \frac{7}{2} & 2 & 0 \end{bmatrix}. \quad its\ Gaus-Jordan\ form \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

So, $a = 0, b = 0, c = 0$. The system has only trivial (i.e. zero) solution. We conclude that S is linearly independent.

Exercise 4.4.8 (Ex. 32. P. 219) Let

$$S = \{(1, 0, 0), (0, 4, 0), (0, 0, -6), (1, 5, -3)\}.$$

Determine, if S is linearly independent or dependent?

Solution: Let

$$c_1(1, 0, 0) + c_2(0, 4, 0) + c_3(0, 0, -6) + c_4(1, 5, -3) = (0, 0, 0).$$

If this equation has only trivial solutions, then it is linealry independent. This equaton gives the following system of linear equations:

$$\begin{array}{cccc} c_1 & & & +c_4 & = 0 \\ & 4c_2 & & 5c_4 & = 0 \\ & & -6c_3 & -3c_4 & = 0 \end{array}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 5 & 0 \\ 0 & 0 & -6 & -3 & 0 \end{bmatrix}. \quad its\ Gaus{-}Jordan\ form \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1.25 & 0 \\ 0 & 0 & 1 & .5 & 0 \end{bmatrix}.$$

Correspondingly:

$$c_1 + c_4 = 0, \quad c_2 + 1.25c_4 = 0, \quad c_3 + .5c_4 = 0.$$

With $c_4 = t$ as parameter, we have

$$c_1 = -t, \quad c_2 = -1.25t, \quad c_3 = .5t, \quad c_4 = t.$$

The equation above has nontrivial (i.e. nonzero) solutions. So, S is linearly dependent.

Theorem 4.4.9 Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_k}\}, k \geq 2$ a set of elements (vectors) in V. Then S is linearly dependent if and only if one of the vectors v_j can be written as a linear combination of the other vectors in S.

Proof. $(\Rightarrow)$: Assume S is linearly dependent. So, the equation

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \mathbf{0}$$

has a nonzero solution. This means, at least one of the c_i is nonzero. Let c_r is the last one, with $c_r \neq 0$. So,

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_r\mathbf{v_r} = \mathbf{0}$$

and

$$\mathbf{v_r} = -\frac{c_1}{c_r}\mathbf{v_1} - \frac{c_2}{c_r}\mathbf{v_2} - \cdots - \frac{c_{r-1}}{c_r}\mathbf{v_{r-1}}.$$

So, $\mathbf{v_r}$ is a linear combination of other vectors and this implication isproved.

$(\Rightarrow)$: to prove the other implication, we assume that $\mathbf{v_r}$ is linear combination of other vectors. So

$$\mathbf{v_r} = (c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_{r-1}\mathbf{v_{r-1}}) + (c_{r+1}\mathbf{v_{r+1}} + \cdots + c_k\mathbf{v_k}).$$

So,

$$(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_{r-1}\mathbf{v_{r-1}}) - \mathbf{v_r} + (c_{r+1}\mathbf{v_{r+1}} + \cdots + c_k\mathbf{v_k}) = \mathbf{0}.$$

The left hand side is a nontrivial (i.e. nozero) linear combination, because $\mathbf{v_r}$ has coefficient -1. Therefore, S is linearly dependent. This completes the proof. ■

4.5 Basis and Dimension

Homework: [Textbook, §4.5 Ex. 1, 3, 7, 11, 15, 19, 21, 23, 25, 28, 35, 37, 39, 41,45, 47, 49, 53, 59, 63, 65, 71, 73, 75, 77, page 231].

The main point *of the section is*

1. *To define basis of a vector space.*
2. *To define dimension of a vector space.*

These are, probably, the two most fundamental concepts regarding vector spaces.

Definition 4.5.1 Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_k}\}$ be a set of elements (vectors)in V. We say that S is a **basis** of V if

1. S spans V and
2. S is linearly independent.

Remark. Here are some some comments about finite and infinite basis of a vector space V :

1. We avoided discussing infinite spanning set S and when an infinite S is linearly independent. We will continue to avoid to do so. (*(1) An infinite set S is said span V, if each element $\mathbf{v} \in V$ is a linear combination of finitely many elements in V. (2) An infinite set S is said to be linearly independent if any finitely subset of S is linearly independent.*)
2. We say that a vector space V is **finite dimensional,** if V has a basis consisting of finitely many elements. Otherwise, we say that V is **infinite dimensional.**
3. The vector space P of all polynomials (with real coefficients) has infinite dimension.

Example 4.5.2 (example 1, p 221) Most standard example of basis is the **standard basis** of $\mathbb{R}^n$.

1. Consider the vector space $\mathbb{R}^2$. Write

$$\mathbf{e_1} = (1, 0), \mathbf{e_2} = (0, 1).$$

Then, $\mathbf{e_1}, \mathbf{e_2}$ form a basis of $\mathbb{R}^2$.

2. Consider the vector space $\mathbb{R}^3$. Write

$$\mathbf{e_1} = (1,0,0), \mathbf{e_2} = (0,1,0), \mathbf{e_2} = (0,0,1).$$

Then, $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ form a basis of $\mathbb{R}^3$.

Proof. First, for any vector $\mathbf{v} = (x_1, x_2, x_3) \in \mathbb{R}^3$, we have

$$\mathbf{v} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3}.$$

So, $\mathbb{R}^3$ is spanned by $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$.

Now, we prove that $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ are linearly independent. So, suppose

$$c_1\mathbf{e_1} + c_2\mathbf{e_2} + c_3\mathbf{e_3} = \mathbf{0} \quad OR \quad (c_1, c_2, c_3) = (0, 0.0).$$

So, $c_1 = c_2 = c_3 = 0$. Therefore, $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ are linearly independent. Hence $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ forms a basis of $\mathbb{R}^3$. The proof is complete.
■

3. More generally, consider vector space $\mathbb{R}^n$. Write

$$\mathbf{e_1} = (1,0,\ldots,0), \mathbf{e_2} = (0,1,\ldots,0), \ldots, \mathbf{e_n} = (0,0,\ldots,1).$$

Then, $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \ldots, \mathbf{e_n}$ form a basis of $\mathbb{R}^n$. The proof will be similar to the above proof. This basis is called the **standard basis** of $\mathbb{R}^n$.

Example 4.5.3 Consider

$$\mathbf{v_1} = (1,1,1), \mathbf{v_2} = (1,-1,1), \mathbf{v_3} = (1,1,-1) \qquad in \quad \mathbb{R}^3.$$

Then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ form a basis for $\mathbb{R}^3$.

Proof. First, we prove that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent. Let

$$c_1\mathbf{v_1}+c_2\mathbf{v_2}+c_3\mathbf{v_3} = \mathbf{0}. \quad OR \quad c_1(1,1,1)+c_2(1,-1,1)+c_3(1,1,-1) = (0,0,0).$$

We have to prove $c_1 = c_2 = c_3 = 0$. The equations give the following system of linear equations:

$$\begin{array}{llll} c_1 & +c_2 & +c_3 & = 0 \\ c_1 & -c_2 & +c_3 & = 0 \\ c_1 & +c_2 & -c_3 & = 0 \end{array}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix} \quad its\ Gauss - Jordan\ form \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So, $c_1 = c_2 = c_3 = 0$ and this estblishes that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent.

Now to show that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ spans $\mathbb{R}^3$, let $\mathbf{v} = (x_1, x_2, x_3)$ be a vector in $\mathbb{R}^3$. We have to show that, we can find c_1, c_2, c_3 such that

$$(x_1, x_2, x_3) = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$$

OR

$$(x_1, x_2, x_3) = c_1(1,1,1) + c_2(1,-1,1) + c_3(1,1,-1).$$

This gives the system of linear equations:

$$\begin{bmatrix} c_1 & +c_2 & +c_3 \\ c_1 & -c_2 & +c_3 \\ c_1 & +c_2 & -c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad OR \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad \textit{has inverse} \quad A^{-1} = \begin{bmatrix} 0 & .5 & .5 \\ .5 & -.5 & 0 \\ .5 & 0 & -.5 \end{bmatrix}.$$

So, the above system has tha solution:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & .5 & .5 \\ .5 & -.5 & 0 \\ .5 & 0 & -.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

So, each vector (x_1, x_2, x_3) is in the span of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. So, they form a basis of $\mathbb{R}^3$. The proof is complete. ■

Reading assignment: Read [Textbook, Examples 1-5, p. 221-224].

Theorem 4.5.4 Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be a basis of V. Then every vector $\mathbf{v}$ in V can be written in one and only one way as a linear combination of vectors in S. *(In other words,* $\mathbf{v}$ *can be written as a unique linear combination of vectors in S.)*

Proof. Since S spans V, we can write $\mathbf{v}$ as a linear combination

$$\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}$$

for scalars $c_1, c_2, \ldots, c_n$. To prove uniqueness, also let

$$\mathbf{v} = d_1\mathbf{v_1} + d_2\mathbf{v_2} + \cdots + d_n\mathbf{v_n}$$

for some other scalars $d_1, d_2, \ldots, d_n$. Subtracting, we have

$$(c_1 - d_1)\mathbf{v_1} + (c_2 - d_2)\mathbf{v_2} + \cdots + (c_n - d_n)\mathbf{v_n} = \mathbf{0}.$$

Since, $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ are also linearly independent, we have

$$c_1 - d_1 = 0, c_2 - d_2 = 0, \ldots, c_n - d_n = 0$$

OR

$$c_1 = d_1, c_2 = d_2, \ldots, c_n = d_n.$$

This completes the proof. ■

Theorem 4.5.5 Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be a basis of V. Then every set of vectors in V containing more than n vectors in V is linearly dependent.

Proof. Suppose $S_1 = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ ne a set of m vectors in V, with $m > n$. We are requaired to prove that the zero vector $\mathbf{0}$ is a nontrivial (i.e. nonzero) linear combination of elements in S_1. Since S is a basis, we have

$$\begin{array}{rllll} \mathbf{u_1} = & c_{11}\mathbf{v_1} & +c_{12}\mathbf{v_2} & +\cdots & +c_{1n}\mathbf{v_n} \\ \mathbf{u_2} = & c_{21}\mathbf{v_1} & +c_{22}\mathbf{v_2} & +\cdots & +c_{2n}\mathbf{v_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{u_m} = & c_{m1}\mathbf{v_1} & +c_{m2}\mathbf{v_2} & +\cdots & +c_{mn}\mathbf{v_n} \end{array}$$

Consider the system of linear equations

$$\begin{array}{lllll} c_{11}x_1 & +c_{22}x_2 & +\cdots & +c_{m1}x_m & =0 \\ c_{12}x_1 & +c_{22}x_2 & +\cdots & +c_{m2}x_m & =0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{1n}x_1 & +c_{2n}x_2 & +\cdots & +c_{mn}x_m & =0 \end{array}$$

which is

$$\begin{bmatrix} c_{11} & c_{22} & \cdots & c_{m1} \\ c_{12} & c_{22} & \cdots & c_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1n} & c_{2n} & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

Since $m > n$, this homegeneous system of linear equations has fewer equations than number of variables. So, the system has a nonzero solution (see [Textbook, theorem 1.1, p 25]). It follows that

$$x_1\mathbf{u_1} + x_2\mathbf{u_2} + \cdots + x_m\mathbf{u_m} = \mathbf{0}.$$

We justify it as follows: First,

$$\begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \dots & \mathbf{u_m} \end{bmatrix} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \dots & \mathbf{v_n} \end{bmatrix} \begin{bmatrix} c_{11} & c_{22} & \cdots & c_{m1} \\ c_{12} & c_{22} & \cdots & c_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1n} & c_{2n} & \cdots & c_{mn} \end{bmatrix}$$

and then

$$x_1\mathbf{u_1} + x_2\mathbf{u_2} + \ldots + x_m\mathbf{u_m} = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \dots & \mathbf{u_m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix}$$

which is

$$= \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \dots & \mathbf{v_n} \end{bmatrix} \begin{bmatrix} c_{11} & c_{22} & \cdots & c_{m1} \\ c_{12} & c_{22} & \cdots & c_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1n} & c_{2n} & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix}$$

which is

$$= \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \dots & \mathbf{v_n} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} = \mathbf{0}.$$

Alternately, at your level the proof will be written more explicitly as follows: $x_1\mathbf{u_1} + x_2\mathbf{u_2} + \ldots + x_m\mathbf{u_m} =$

$$\sum_{j=i}^{m} x_j\mathbf{u_j} = \sum_{j=1}^{m} x_j \left(\sum_{i=1}^{n} c_{ij}\mathbf{v_i} \right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} c_{ij}x_j \right) \mathbf{v_i} = \sum_{i=1}^{n} 0\mathbf{v_i} = \mathbf{0}.$$

The proof is complete. ■

Theorem 4.5.6 Suppose V is a vector space and V has a basis with n vectors. Then, every basis has n vectors.

Proof. Let

$$S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\} \quad and \quad S_1 = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$$

be two bases of V. Since S is a basis and S_1 is linearly independent, by theorem 4.5.5, we have $m \leq n$. Similarly, $n \leq m$. So, $m = n$. The proof is complete. ■

Definition 4.5.7 If a vector space V has a basis consisting of n vectors, then we say that dimension of V is n. We also write $\dim(V) = n$. If $V = \{\mathbf{0}\}$ is the zero vector space, then the dimension of V is defined as zero.

(*We say that the dimension of V is equal to the 'cardinality' of any basis of V. The word 'cardinality' is used to mean 'the number of elements' in a set.*)

Theorem 4.5.8 Suppose V is a vector space of dimension n.

1. Suppose $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a set of n linearly independent vectors. Then S is basis of V.

2. Suppose $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a set of n vectors. If S spans V, then S is basis of V.

Remark. The theorem 4.5.8 means that, if dimension of V matches with the number of (i.e. 'cardinality' of) S, then to check if S is a basis of V or not, you have check only one of the two required prperties (1) indpendece or (2) spannning.

Example 4.5.9 Here are some standard examples:

1. We have $\dim(\mathbb{R}) = 1$. This is because $\{1\}$ forms a basis for $\mathbb{R}$.

2. We have $\dim(\mathbb{R}^2) = 2$. This is because the standard basis
$$\mathbf{e_1} = (1, 0), \mathbf{e_2} = (0, 1)$$
consist of two elements.

3. We have $\dim(\mathbb{R}^3) = 3$. This is because the standard basis
$$\mathbf{e_1} = (1, 0, 0), \mathbf{e_2} = (0, 1, 0), \mathbf{e_3} = (0, 0, 1)$$
consist of three elements.

4. Mor generally, $\dim(\mathbb{R}^n) = n$. This is because the standard basis
$$\mathbf{e_1} = (1, 0, 0, \ldots, 0), \mathbf{e_2} = (0, 1, 0, \ldots, 0), \ldots, \mathbf{e_n} = (0, 0, \ldots, 1)$$
consist of n elements.

5. The dimension of the vector space $\mathbb{M}_{m,n}$ of all $m \times n$ matrices is mn. Notationally, $\dim(\mathbb{M}_{m,n}) = mn$. To see this, let $\mathbf{e_{ij}}$ be the $m \times n$ matrix whose $(i, j)^{th}-$entry is 1 and all the rest of the entries are zero. Then,
$$S = \{\mathbf{e_{ij}} : i = 1, 2, \ldots, m; j1, 2, \ldots, n\}$$
forms a basis of $\mathbb{M}_{m,n}$ and S has mn elements.

6. Also recall, if a vector space V does not have a finite basis, we say V is inifinite dimensional.
   (a) The vector space $\mathbb{P}$ of all polynomials (with real coefficients) has infinite dimension.
   (b) The vector space $C(\mathbb{R})$ of all continuous real valued functions on real line $\mathbb{R}$ has infinite dimension.

Exercise 4.5.10 (Ex. 4 (changed), p. 230) Write down the standard basis of the vector space $\mathbb{M}_{3,2}$ of all $3 \times 2-$matrices with real entires.

Solution: Let $\mathbf{e_{ij}}$ be the $3 \times 2-$matrix, whose $(i,j)^{th}-$entry is 1 and all other entries are zero. Then,

$$\{\mathbf{e_{11}}, \mathbf{e_{12}}, \mathbf{e_{21}}, \mathbf{e_{22}}, \mathbf{e_{31}}, \mathbf{e_{32}}\}$$

forms a basis of $\mathbb{M}_{3,2}$. More explicitly,

$$\mathbf{e_{11}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{e_{12}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{e_{21}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\mathbf{e_{22}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{e_{31}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{e_{33}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to verify that these vectors in $\mathbb{M}_{32}$ spans $\mathbb{M}_{32}$ and are linearly independent. So, they form a basis.

Exercise 4.5.11 (Ex. 8. p. 230) Explain, why the set $S = \{(-1,2), (1,-2), (2,4)\}$ is not a basis of $\mathbb{R}^2$?

Solution: Note

$$(-1,2) + (1,-2) + 0(2,4) = (0,0).$$

So, these three vectors are not linearly independent. So, S is not a basis of $\mathbb{R}^2$.

Alternate argument: We have $\dim(\mathbb{R}^2) = 2$ and S has 3 elements. So, by theorem 4.5.6 above S cannot be a basis.

Exercise 4.5.12 (Ex. 16. p. 230) Explain, why the set

$$S = \{(2, 1, -2), (-2, -1, 2), (4, 2, -4)\}$$

is not a basis of $\mathbb{R}^3$?

Solution: Note

$$(4, 2, -4) = (2, 1, -2) - (-2, -1, 2)$$

OR

$$(2, 1, -2) - (-2, -1, 2) - (4, 2, -4) = (0, 0, 0).$$

So, these three vectors are linearly dependent. So, S is not a basis of $\mathbb{R}^3$.

Exercise 4.5.13 (Ex. 24. p. 230) Explain, why the set

$$S = \{6x - 3, 3x^2, 1 - 2x - x^2\}$$

is not a basis of $\mathbb{P}_2$?

Solution: Note

$$1 - 2x - x^2 = -\frac{1}{3}(6x - 3) - \frac{1}{3}(3x^2)$$

OR

$$(1 - 2x - x^2) + \frac{1}{3}(6x - 3) + \frac{1}{3}(3x^2) = \mathbf{0}.$$

So, these three vectors are linearly dependent. So, S is not a basis of $\mathbb{P}_2$.

Exercise 4.5.14 (Ex. 36,p.231) Determine, whether

$$S = \{(1, 2), (1, -1)\}$$

is a basis of $\mathbb{R}^2$ or not?

Solution: We will show that S is linearly independent. Let

$$a(1,2) + b(1,-1) = (0,0).$$

Then

$$a + b = 0, \quad and \quad 2a - b = 0.$$

Solving, we get $a = 0, b = 0$. So, these two vectors are linearly independent. We have $\dim(\mathbb{R}^2) = 2$. Therefore, by theorem 4.5.8, S is a basis of $\mathbb{R}^2$.

Exercise 4.5.15 (Ex. 40. p.231) Determine, whether

$$S = \{(0,0,0), (1,5,6), (6,2,1)\}$$

is a basis of $\mathbb{R}^3$ or not?

Solution: We have

$$1.(0,0,0) + 0.(1,5,6) + 0.(6,2,1) = (0,0,0).$$

So, S is linearly dependent and hence is not a basis of $\mathbb{R}^3$.

Remark. *In fact, any subset S of a vector space V that contains* $\mathbf{0}$ *is linearly dependent.*

Exercise 4.5.16 (Ex. 46. p.231) Determine, whether

$$S = \left\{4t - t^2, 5 + t^3, 3t + 5, 2t^3 - 3t^2\right\}$$

is a basis of $\mathbb{P}_3$ or not?

Solution: Note the standard basis

$$\left\{1, t, t^2, t^3\right\}$$

of $\mathbb{P}_3$ has four elements. So, $\dim(\mathbb{P}_3) = 4$. Because of theorem 4.5.8, we will try to check, if S is linearly independent or not. So, let

$$c_1(4t - t^2) + c_2(5 + t^3) + c_3(3t + 5) + c_4(2t^3 - 3t^2) = 0$$

for some scalars c_1, c_2, c_3, c_4. If we simplify, we get

$$(5c_2 + 5c_3) + (4c_1 + 3c_3)t + (-c_1 - 3c_4)t^2 + (c_2 + 2c_4)t^3 = 0$$

Recall, a polynomial is zero if and only if all the coefficients are zero. So, we have

$$\begin{array}{rrrrl} & 5c_2 & +5c_3 & & = 0 \\ 4c_1 & & +3c_3 & & = 0 \\ -c_1 & & & -3c_4 & = 0 \\ & c_2 & & +2c_4 & = 0 \end{array}$$

The augmented matrix is

$$\begin{bmatrix} 0 & 5 & 5 & 0 & 0 \\ 4 & 0 & 3 & 0 & 0 \\ -1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix} \quad its\, Gauss{-}Jordan\, form \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Therefore, $c_1 = c_2 = c_3 = c_4 = 0$. Hence S is linearely independent. So, by theorem 4.5.8, S is a basis of $\mathbb{P}_3$.

Exercise 4.5.17 (Ex. 60. p.231) Determine the dimension of $\mathbb{P}_4$.

Solution: Recall, $\mathbb{P}_4$ is the vector space of all polynomials of degree ≤ 4. We claim that that

$$S = \{1, t, t^2, t^3, t^4\}$$

is a basis of $\mathbb{P}_4$. Clearly, any polynomial in $\mathbb{P}_4$ is a linear combination of elements in S. So, S spans $\mathbb{P}_4$. Now, we prove that S is linearly independent. So, let

$$c_0 1 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 = 0.$$

Since a nonzero polynomial of degree 4 can have at most four roots, it follows $c_0 = c_1 = c_2 = c_3 = c_4 = 0$. So, S is a basis of $\mathbb{P}_4$ and $\dim(\mathbb{P}_4) = 5$.

Exercise 4.5.18 (Ex. 62. p.231) Determine the dimension of $\mathbb{M}_{32}$.

Solution: In exercise 4.5.10, we established that

$$S = \{\mathbf{e_{11}}, \mathbf{e_{12}}, \mathbf{e_{21}}, \mathbf{e_{22}}, \mathbf{e_{31}}, \mathbf{e_{32}}\}$$

is a basis of $\mathbb{M}_{3,2}$. So, $\dim(\mathbb{M}_{32}) = 6$.

Exercise 4.5.19 (Ex. 72. p.231) Let

$$W = \{(t, s, t) : s, t \in \mathbb{R}\}.$$

Give a geometric description of W, find a basis of W and determine the dimension of W.

Solution: First note that W is closed under addition and scalar multiplication. So, W is a subspace of $\mathbb{R}^3$. Notice, there are two parameters s, t in the description of W. So, W can be described by $x = z$. Therefore, W represents the plane $x = z$ in $\mathbb{R}^3$.

I suggest (guess) that

$$\mathbf{u} = (1, 0, 1), \mathbf{v} = (0, 1, 0)$$

will form a basis of W. To see that they are mutually linearly independent, let

$$a\mathbf{u} + b\mathbf{v} = (0.0.0); \quad OR \quad (a, b, a) = (0.0.0).$$

So, $a = 0, b = 0$ and hence they are linearly independent. To see that they span W, we have

$$(t, s, t) = t\mathbf{u} + s\mathbf{v}.$$

So, $\{\mathbf{u}, \mathbf{v}\}$ form a basis of W and $\dim(W) = 2$.

Exercise 4.5.20 (Ex. 74. p.232) Let

$$W = \{(5t, -3t, t, t) : t \in \mathbb{R}\}.$$

Fnd a basis of W and determine the dimension of W.

Solution: First note that W is closed under addition and scalar multiplication. So, W is a subspace of $\mathbb{R}^4$. Notice, there is only parameters t in the description of W. (So, I expect that $\dim(W) = 1$. I suggest (guess)

$$\mathbf{e} = \{(5, -3, 1, 1)\}$$

is a basis of W. This is easy to check. So, $\dim(W) = 1$.

4.6 Rank of a matrix and SoLE

Homework: [Textbook, §4.6 Ex. 7, 9, 15, 17, 19, 27, 29, 33, 35, 37, 41, 43, 47, 49, 57, 63].

Main topics in this section *are to define*

1. *We define row space of a matrix A and the column space of a matrix A.*
2. *We define the rank of a matrix,*
3. *We define* **nullspace** $N(A)$ *of a homoheneous system* $A\mathbf{x} = \mathbf{0}$ *of linear equations. We also define the* **nullity** *of a matrix A.*

Definition 4.6.1 Let $A = [a_{ij}]$ be an $m \times n$ matrix.

1. The $n-$tuples corresponding to the rows of A are called **row vectors** of A.
2. Similarly, the $m-$tuples corresponding to the columns of A are called **column vectors** of A.
3. The **row space** of A is the subspace of $\mathbb{R}^n$ spanned by row vectors of A.
4. The **column space** of A is the subspace of $\mathbb{R}^m$ spanned by column vectors of A.

Theorem 4.6.2 Suppose A, B are two $m \times n$ matrices. If A is row-equivalent of B then row space of A is equal to the row space of B.

Proof. This follows from the way row-equivalence is defined. Since B is rwoequivalent to A, rows of B are obtained by (a series of) scalar multiplication and addition of rows of A. So, it follows that row vectors of B are in the row space of A. Therefore, the subspace spanned by row vectors of B is contained in the row space of A. So, the row space of B is contained in the row space of A. Since A is row-equivalent of B, it also follows the B is row-equivalent of A. (*We say that the 'relationship' of being 'row-equivalent' is reflexive.*) Therefore, by the same argumen, the row space of A is contained in the row space of B. So, they are equal. The proof is complete. ■

Theorem 4.6.3 Suppose A is an $m \times n$ matrix and B is row-equivalent to A and B is in row-echelon form. Then the nonzero rows of B form a basis of the row space of A.

Proof. From theorem 4.6.2, it follows that row space of A and B are some. Also, a basis of the row space of B is given by the nonzero rows of B. The proof is complete. ■

Theorem 4.6.4 Suppose A is an $m \times n$ matrix. Then the row space and column space of A have same dimension.

Proof. (*You can skip it, I will not ask you to prove this.*) Write

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}$ denote the row vectors of A and $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$ denote the column vectors of A. Suppose that the row space of A has dimension r and

$$S = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_r}\}$$

is a basis of the row space of A. Also, write

$$\mathbf{b_i} = (b_{i1}, b_{i2}, \ldots, b_{in}).$$

We have

$$\begin{array}{rccccc} \mathbf{v_1} = & c_{11}\mathbf{b_1} & +c_{12}\mathbf{b_2} & +\cdots & +c_{1r}\mathbf{b_r} \\ \mathbf{v_2} = & c_{21}\mathbf{b_1} & +c_{22}\mathbf{b_2} & +\cdots & +c_{2r}\mathbf{b_r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{v_m} = & c_{m1}\mathbf{b_1} & +c_{m2}\mathbf{b_2} & +\cdots & +c_{mr}\mathbf{b_r} \end{array}$$

Looking at the first entry of each of these m equations, we have

$$\begin{array}{rccccc} a_{11} = & c_{11}b_{11} & +c_{12}b_{21} & \cdots & +c_{1r}b_{r1} \\ a_{21} = & c_{21}b_{11} & +c_{22}b_{21} & \cdots & +c_{2r}b_{r1} \\ a_{31} = & c_{31}b_{11} & +c_{32}b_{21} & \cdots & +c_{3r}b_{r1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} = & c_{m1}b_{11} & +c_{m2}b_{21} & \cdots & +c_{mr}b_{r1} \end{array}$$

Let $\mathbf{c_i}$ denote the i^{th} column of the matrix $C = [c_{ij}]$. So, it follows from these m equations that

$$\mathbf{u_1} = b_{11}\mathbf{c_1} + b_{21}\mathbf{c_2} + \cdots + b_{r1}\mathbf{c_r}.$$

Similarly, looking at the j^{th} entry of the above set of equations, we have

$$\mathbf{u_j} = b_{1j}\mathbf{c_1} + b_{2j}\mathbf{c_2} + \cdots + b_{rj}\mathbf{c_r}.$$

So, all the columns $\mathbf{u_j}$ of A are in $span(\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r})$. Therefore, the column space of A is contained in $span(\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r})$. It follows from this that the rank of the column space of A has dimension $\leq r =$ rank of the row space of A. So,

$$\dim(column\ space\ of\ A) \leq \dim(row\ space\ of\ A).$$

Similarly,

$$\dim(row\ space\ of\ A) \leq \dim(column\ space\ of\ A).$$

So, they are equal. The proof is complete. ■

Definition 4.6.5 Suppose A is an $m \times n$ matrix. The dimension of the row space (equivalently, of the column space) of A is called the **rank** of A and is denoted by $rank(A)$.

Reading assignment: Read [Textbook, Examples 2-5, p. 234-].

4.6.1 The Nullspace of a matrix

Theorem 4.6.6 Suppose A is an $m \times n$ matrix. Let $N(A)$ denote the set of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$. Notationally:

$$N(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

Then $N(A)$ is a a subspace of $\mathbb{R}^n$ and is called the **nullspace** of A. The dimension of $N(A)$ is called the **nullity** of A. Notationally:

$$nullity(A) := \dim(N(A)).$$

Proof. First, $N(A)$ is nonempty, because $\mathbf{0} \in N(A)$. By theorem 4.3.3, we need only to check that $N(A)$ is closed under addition and scalar multiplication. Suppose $\mathbf{x}, \mathbf{y} \in N(A)$ and c is a scalar. Then

$$A\mathbf{x} = \mathbf{0}, \quad A\mathbf{y} = \mathbf{0}, \quad so \quad A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

So, $\mathbf{x} + \mathbf{y} \in N(A)$ and $N(A)$ is closed under addition. Also

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c\mathbf{0} = \mathbf{0}.$$

Therefore, $c\mathbf{x} \in N(A)$ and $N(A)$ is closed under scalar multiplication.

Theorem 4.6.7 Suppose A is an $m \times n$ matrix. Then

$$rank(A) + nullity(A) = n.$$

That means, $\dim(N(A)) = n - rank(A)$.

Proof. Let $r = rank(A)$. Let B be a matrix row equivalent to A and B is in Gauss-Jordan form. So, only the first r rows of B are nonzero. Let B' be the matrix formed by top r (i.e. nonzero) rows of B. Now,

$$rank(A) = rank(B) = rank(B'), \quad nullity(A) = nullity(B) = nullity(B').$$

So, we need to prove $rank(B') + nullity(B') = n$. Switching columns of B' would only mean re-labeling the variables (like $x_1 \mapsto x_1, x_2 \mapsto x_3, x_3 \mapsto x_2$). In this way, we can write $B' = [I_r, C]$, where C is a $r \times n - r$ matrix and corresponds to the variables, $x_{r+1}, \ldots, x_n$. The homogeneous system corresponding to B' is given by:

$$\begin{array}{ccccccccc}
x_1 & & \cdots & & +c_{11}x_{r+1} & +c_{12}x_{r+2} & +\cdots & +c_{1,n-r}x_n & = 0 \\
 & x_2 & \cdots & & +c_{21}x_{r+1} & +c_{22}x_{r+2} & +\cdots & +c_{2,n-r}x_n & = 0 \\
 & & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & \cdots & x_r & +c_{r1}x_{r+1} & +c_{r2}x_{r+2} & +\cdots & +c_{r,n-r}x_n & = 0
\end{array}$$

The solution space $N(B')$ has $n - r$ papameters. A basis of $N(B')$ is given by

$$S = \{\mathbf{E_{r+1}}, \mathbf{E_{r+2}}, \ldots, \mathbf{E_n}\}$$

where

$$\mathbf{E_{r+1}} = -\left(c_{11}e_1 + c_{21}e_2 + \cdots + c_{r1}e_r\right) + e_{r+1} \quad \textit{so on}$$

and $\mathbf{e_i} \in \mathbb{R}^n$ is the vector with 1 at the i^{th} place and 0 elsewhere. So, $nullity(B') = cardinality(S) = n - r$. The proof is complete. ■

Reading assignment: Read [Textbook, Examples 6, 7, p. 241-242].

4.6.2 Solutionf of SoLE

Given a system of linear equations $A\mathbf{x} = \mathbf{b}$, where A is an $m \times n$ matrix, we have the following:

1. Corresponding to such a system $A\mathbf{x} = \mathbf{b}$, there is a homogeneous system $A\mathbf{x} = \mathbf{0}$.
2. The set of solutions $N(A)$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.
3. In contrast, if $\mathbf{b} \neq \mathbf{0}$, the set of solutions of $A\mathbf{x} = \mathbf{b}$ is not a subspace. This is because $\mathbf{0}$ is not a solution of $A\mathbf{x} = \mathbf{b}$.
4. The system $A\mathbf{x} = \mathbf{b}$ may have many solution. Let $\mathbf{x_p}$ denote a PARTICULAR one such solutions of $A\mathbf{x} = \mathbf{b}$.
5. The we have

   Theorem 4.6.8 *Every solution of the system $A\mathbf{x} = \mathbf{b}$ can be written as*

   $$\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$$

   where $\mathbf{x_h}$ is a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

   Proof. Suppose $\mathbf{x}$ is any solution of $A\mathbf{x} = \mathbf{b}$. We have

   $$A\mathbf{x} = \mathbf{b} \quad \textit{and} \quad A\mathbf{x_p} = \mathbf{b}.$$

Write $\mathbf{x_h} = \mathbf{x} - \mathbf{x_p}$. Then

$$A\mathbf{x_h} = A(\mathbf{x} - \mathbf{x_p}) = A\mathbf{x} - A\mathbf{x_p} = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

So, $\mathbf{x_h}$ is a solution of the homogeneoud system $A\mathbf{x} = \mathbf{0}$ and

$$\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}.$$

The proof is complete. ■

Theorem 4.6.9 A system $A\mathbf{x} = \mathbf{b}$is consistent if and only if $\mathbf{b}$ is in the column space of A.

Proof. Easy. It is, in fact, interpretation of the matrix multiplication $A\mathbf{x} = \mathbf{b}$.

Reading assignment: Read [Textbook, Examples 8,9, p. 244-245].

Theorem 4.6.10 Suppose A is a square matrix of size $n \times n$. Then the following conditions are equivalent:

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has unique solution for every $m \times 1$ matrix $\mathbf{b}$.
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. A is row equivalent to the identity matrix I_n.
5. $\det(A) \neq 0$.
6. $Rank(A) = n$.
7. The n row vectors of A are linearly independent.
8. The n column vectors of A are linearly independent.

Exercise 4.6.11 (Ex. 8, p. 246) Let

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 10 & 6 \\ 8 & -7 & 5 \end{bmatrix}.$$

(a) Find the rank of the matrix A. (b) Find a basis of the row space of A, (c) Find a basis of the column space of A.

Solution: First, the following is the row Echelon form of this matrix (use TI):

$$B = \begin{bmatrix} 1 & -.875 & .625 \\ 0 & 1 & .2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rank of A is equal to the number of nonzero rows of B. So, $rank(A) = 2$.

A basis of the row space of A is given by the nonzero rwos of B. So,

$$\mathbf{v_1} = (1, -.875, .625) \quad and \quad \mathbf{v_2} = (0, 1, .2)$$

form a basis of the row space of A.

The column space of A is same as the row space of the transpose A^T. We have

$$A^T = \begin{bmatrix} 2 & 5 & 8 \\ -3 & 10 & -7 \\ 1 & 6 & 5 \end{bmatrix}.$$

The following is the row Echelon form of this matrix (use TI):

$$C = \begin{bmatrix} 1 & -\frac{10}{3} & \frac{7}{3} \\ 0 & 1 & 0.2857 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis of the column space of A is given by the nonzero rows of C, (to be written as column):

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ -\frac{10}{3} \\ \frac{7}{3} \end{bmatrix}, \quad \mathbf{u_2} = \begin{bmatrix} 0 \\ 1 \\ 0.2857 \end{bmatrix}.$$

Exercise 4.6.12 (Ex. 16, p. 246) Let

$$S = \{(1, 2, 2), (-1, 0, 0), (1, 1, 1)\} \subseteq \mathbb{R}^3.$$

Find a basis of of the subspace spanned by S.

Solution: We write these rows as a matrix:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Now the row space of A will be the same as the subspace spanned by S. So, we will find a basis of the row space of A. Use TI and we get the row Echelon form of A is given by

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, a basis is:

$$\mathbf{u_1} = (1, 2, 2), \quad \mathbf{u_2} = (0, 1, 1).$$

Remark. The answers regrading bases would not be unique. The following will also be a basis of this space:

$$\mathbf{v_1} = (1, 2, 2), \quad \mathbf{v_2} = (1, 0, 0).$$

Exercise 4.6.13 (Ex. 20, p. 246) Let

$$S = \{(2,5,-3,-2),(-2,-3,2,-5),(1,3,-2,2),(-1,-5,3,5)\} \subseteq \mathbb{R}^4.$$

Find a basis of of the subspace spanned by S.

Solution: We write these rows as a matrix:

$$A = \begin{bmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ -1 & -5 & 3 & 5 \end{bmatrix}.$$

Now the row space of A will be the same as the subspace spanned by S. So, we will find a basis of the row space of A.

Use TI and we get the row Echelon form of A is given by

$$B = \begin{bmatrix} 1 & 2.5 & -1.5 & -1 \\ 0 & 1 & -0.6 & -1.6 \\ 0 & 0 & 1 & -19 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, a basis is:

$$\{\mathbf{u_1} = (1, 2.5, -1.5, -1), \quad \mathbf{u_2} = (0, 1, -0.6, -1.6), \quad \mathbf{u_3} = (0, 0, 1, -19)\}.$$

Exercise 4.6.14 (Ex. 28, p. 247) Let

$$A = \begin{bmatrix} 3 & -6 & 21 \\ -2 & 4 & -14 \\ 1 & -2 & 7 \end{bmatrix}.$$

Find the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$.

Solution: *Step-1: Find rank of A* : Use TI, the row Echelon form of A is

$$B = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, the number of nonzero rows of B is $rank(A) = 1$.

Step-2: By theorem 4.6.7, we have

$$rank(A) + nullity(A) = n = 3, \qquad so \quad nullity(A) = 3 - 1 = 2.$$

That means that the solution space has dimension 2.

Exercise 4.6.15 (Ex. 32, p. 247) Let

$$A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 2 & -1 & 1 & 1 \\ 4 & 2 & 1 & 1 \\ 0 & 4 & 2 & 0 \end{bmatrix}.$$

Find the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$.

Solution: *Step-1: Find rank of A* : Use TI, the row Echelon form of A is

$$B = \begin{bmatrix} 1 & .5 & .25 & .25 \\ 0 & 1 & .5 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So, the number of nonzero rows of B is $rank(A) = 4$.

Step-2: By theorem 4.6.7, we have

$$rank(A) + nullity(A) = n = 4, \qquad so \quad nullity(A) = 4 - 4 = 0.$$

That means that the solution space has dimension 0. This also means that the the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Exercise 4.6.16 (Ex. 38 (edited), p. 247) Consider the homogeneous system

$$\begin{array}{rrrrl} 2x_1 & +2x_2 & +4x_3 & -2x_4 & = 0 \\ x_1 & +2x_2 & +x_3 & +2x_4 & = 0 \\ -x_1 & +x_2 & +4x_3 & -x_4 & = 0 \end{array}$$

Find the dimension of the solution space and give a basis of the same.

Solution: We follow the following steps:

1. First, we write down the coefficient matrix:

$$A = \begin{bmatrix} 2 & 2 & 4 & -2 \\ 1 & 2 & 1 & 2 \\ -1 & 1 & 4 & -1 \end{bmatrix}$$

2. Use TI, the Gauss-Jordan for of the matrix is

$$B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

3. The rank of A is number of nonzero rows of B. So,

$$rank(A) = 3, \quad by\, thm.\, 4.6.7, \quad nullity(A) = n - rank(A) = 4 - 3 = 1.$$

   So, the solution space has dimension 1.

4. To find the solution space, we write down the homogeneous system corresponding to the coeeficient matrix B. So, we have

$$\begin{array}{cccrl} x_1 & & & -x_4 & = 0 \\ & x_2 & & +2x_4 & = 0 \\ & & x_3 & -x_4 & = 0 \end{array}$$

5. Use $x_4 = t$ as parameter and we have

$$x_1 = t, \quad x_2 = -2t, \quad x_3 = t, \quad x_4 = t.$$

6. So the solution space is given by

$$\{(t, -2t, t, t) : t \in \mathbb{R}\}.$$

7. A basis is obtained by substituting $t = 1$. So

$$\mathbf{u} = (1, -2, 1, 1)$$

forms a basis of the solution space.

Exercise 4.6.17 (Ex. 39, p. 247) Consider the homogeneous system

$$\begin{array}{rrrrl} 9x_1 & -4x_2 & -2x_3 & -20x_4 & = 0 \\ 12x_1 & -6x_2 & -4x_3 & -29x_4 & = 0 \\ 3x_1 & -2x_2 & & -7x_4 & = 0 \\ 3x_1 & -2x_2 & -x_3 & -8x_4 & = 0 \end{array}$$

Find the dimension of the solution space and give a basis of the same.

Solution: We follow the following steps:

1. First, we write down the coefficient matrix:

$$A = \begin{bmatrix} 9 & -4 & -2 & -20 \\ 12 & -6 & -4 & -29 \\ 3 & -2 & 0 & -7 \\ 3 & -2 & -1 & -8 \end{bmatrix}$$

2. Use TI, the Gauss-Jordan for of the matrix is

$$B = \begin{bmatrix} 1 & 0 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3. The rank of A is number of nonzero rows of B. So,

$$rank(A) = 3, \quad by\, thm.\, 4.6.7, \quad nullity(A) = n - rank(A) = 4 - 3 = 1.$$

So, the solution space has dimension 1.

4. To find the solution space, we write down the homogeneous system corresponding to the coeeficient matrix B. So, we have

$$\begin{array}{ccccc} x_1 & & & -\frac{4}{3}x_4 & = 0 \\ & x_2 & & +1.5x_4 & = 0 \\ & & x_3 & +x_4 & = 0 \\ & & & 0 & = 0 \end{array}$$

5. Use $x_4 = t$ as parameter and we have

$$x_1 = \frac{4}{3}t, \quad x_2 = -1.5t, \quad x_3 = -t, \quad x_4 = t.$$

6. So the solution space is given by

$$\left\{ (\frac{4}{3}t, -1.5t, -t, t) : t \in \mathbb{R} \right\}.$$

7. A basis is obtained by substituting $t = 1$. So

$$\mathbf{u} = (\frac{4}{3}, -1.5, -1, 1)$$

forms a basis of the solution space.

Exercise 4.6.18 (Ex. 42, p. 247) Consider the system of equations

$$\begin{array}{rrrrl} 3x_1 & -8x_2 & +4x_3 & & = 19 \\ & -6x_2 & +2x_3 & +4x_4 & = 5 \\ 5x_1 & & +22x_3 & +x_4 & = 29 \\ x_1 & -2x_2 & +2x_3 & & = 8 \end{array}$$

Determine, if this system is consistent. If yes, write the solution in the form $\mathbf{x} = \mathbf{x_h} + \mathbf{x_p}$ where $\mathbf{x_h}$ is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x_p}$ is a particular solution.

Solution: We follow the following steps:

1. To find a particular solution, we write the augmented matrix of the nonhomogeneous system:

$$\begin{bmatrix} 3 & -8 & 4 & 0 & 19 \\ 0 & -6 & 2 & 4 & 5 \\ 5 & 0 & 22 & 1 & 29 \\ 1 & -2 & 2 & 0 & 8 \end{bmatrix}$$

The Gauss-Jordan form of the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -.5 & 0 \\ 0 & 0 & 1 & .5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The last row suggests $0 = 1$. So, the system is not consistents.

Exercise 4.6.19 (Ex. 44, p. 247) Consider the system of equations

$$\begin{array}{rrrl} 2x_1 & -4x_2 & +5x_3 & = 8 \\ -7x_1 & +14x_2 & +4x_3 & = -28 \\ 3x_1 & -6x_3 & +x_3 & = 12 \end{array}$$

Determine, if this system is consistent.If yes, write the solution in the form $\mathbf{x} = \mathbf{x_h} + \mathbf{x_p}$ where $\mathbf{x_h}$ is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x_p}$ is a particular solution.

Solution: We follow the following steps:

1. First, the augmented matrix of the system is
$$\begin{bmatrix} 2 & -4 & 5 & 8 \\ -7 & 14 & 4 & -28 \\ 3 & -6 & 1 & 12 \end{bmatrix}.$$
Its Gauss-Jordan form is
$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
This corresponds to they system
$$\begin{array}{cccc} x_1 & -2x_2 & & = 4 \\ & & x_3 & = 0 \\ & & 0 = & 0 \end{array}.$$
The last row indicates that the system is consistent. We use $x_2 = t$ as a paramater and we have
$$x_1 = 4 + 2t, \quad x_2 = t, \quad x_3 = 0.$$
Thaking $t = 0$, a particular solutions is
$$\mathbf{x_p} = (4, 0, 0).$$

2. Now, we proceed to find the solution of the homogeneous system
$$\begin{array}{cccc} 2x_1 & -4x_2 & +5x_3 & = 0 \\ -7x_1 & +14x_2 & +4x_3 & = 0 \\ 3x_1 & -6x_3 & +x_3 & = 0 \end{array}$$

(a) The coefficient matrix

$$A = \begin{bmatrix} 2 & -4 & 5 \\ -7 & 14 & 4 \\ 3 & -6 & 1 \end{bmatrix}.$$

(b) Its Gauss-Jordan form is

$$B = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) The homogeneous system corresponding to B is

$$\begin{array}{rrrl} x_1 & -2x_2 & & = 0 \\ & & x_3 & = 0 \\ & & 0 & = 0 \end{array}$$

(d) We use $x_2 = t$ as a paramater and we have

$$x_1 = 2t, \quad x_2 = t, \quad x_3 = 0.$$

(e) So, in parametrix form

$$\mathbf{x_h} = (2t, t, 0).$$

3. Final answer is: With t as parameter, any solutions can be written as

$$\mathbf{x} = \mathbf{x_h} + \mathbf{x_p} = (2t, t, 0) + (4, 0, 0).$$

Exercise 4.6.20 (Ex. 50, p. 247) Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad and \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Determine, if $\mathbf{b}$ is in the column space of A.

Solution: The question means, whether the system $A\mathbf{x} = \mathbf{b}$ has a solutions (i.e. *is consistent*).

Accordingly, the augmented matrix of this system $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ -1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The Gauss-Jordan form of this matrix is i

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The last row indicates that the system is not consistent. So, $\mathbf{b}$ is not in the column space of A.