$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$ Characteristic Equation $\det(A-\lambda I)=0$ ①

$$\det \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} = 0 \longrightarrow \text{matrix } *$$

$$-\lambda(2-\lambda)(3-\lambda) - 2(-(2-\lambda)) = 0$$
$$(2-\lambda)(-\lambda(3-\lambda)) + 2(2-\lambda) = 0$$
$$(2-\lambda)(-\lambda(3-\lambda)+2) = 0$$
$$(2-\lambda)(\lambda^2 - 3\lambda + 2) = 0 \longrightarrow \text{quadratic polynomial factoring}$$
$$(2-\lambda)(\lambda-1)(\lambda-2) = 0$$

Eignvalues:- $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 2$ repeated.

$\boxed{\text{For } \lambda_1, \lambda_3}$ we solve $(A-\lambda I)x = 0$ (λ here $=2$)

put $\lambda = 2$ in matrix $*$

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{aligned} -2x_1 - 2x_3 &= 0 \\ x_1 + x_3 &= 0 \\ x_1 + x_3 &= 0 \end{aligned}$$

$$\Rightarrow x_1 = -x_3$$
$$x_2 = \text{free}$$

Choose $\begin{cases} x_2 = 0 \\ x_3 = 1 \end{cases}$ $\therefore x_1 = -1$ $\quad P_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ } we need two eigenvectors for λ_1, λ_3.

Choose $\begin{cases} x_2 = 1 \\ x_3 = 0 \end{cases}$ $\therefore x_1 = 0$

$\boxed{\text{For } \lambda_2 = 1}$

$$(A-\lambda I)x = 0 \quad \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -x_1 - 2x_3 &= 0 \rightarrow 1 \\ x_1 + x_2 + x_3 &= 0 \rightarrow 2 \\ x_1 + 2x_3 &= 0 \rightarrow 3 \end{aligned} \Rightarrow \boxed{x_1 = -2x_3} \text{ from } 1 \& 3$$

②

from 2 $\quad x_1 = -x_2 - x_3$

$$-2x_3 = -x_2 - x_3$$

$$-x_3 = -x_2$$

$$\therefore \boxed{x_2 = x_3}$$

$x_1 = -2x_3$

$x = x_3$

x_3 free

choose $x_3 = 1 \quad \therefore x_2 = 1, \; x_1 = -2$

$$P_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \qquad \therefore P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & +1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \overbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{B}$$

Eigenvalues

B is similar to ~~B~~ A.

Complex dot product.

Choose

$u = (2i, 3, 4+i)$

$v = (1+i, -4i, 5)$

$u \cdot v = u \cdot \bar{v} = (2i, 3, 4+i) \cdot (1-i, 4i, 5)$

$= (2i)(1-i) + (3)(4i) + (4+i)(5)$

$= 2i + 2 + 12i + 20 + 5i$

$\boxed{u \cdot v = 22 + 19i}$

$v \cdot u = v \cdot \bar{u} = (1+i, -4i, 5)(-2i, 3, 4-i)$

$= -2i + 2 - 12i + 20 - 5i = 22 - 19i$

$\boxed{v \cdot u = 22 - 19i = \overline{u \cdot v}}$.

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if $A = \begin{bmatrix} 2i & i \\ 1 & 1+i \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1+2i \\ -2i & 3 \end{bmatrix}$

Show $\overline{AB} = \bar{A}\bar{B}$.

$\bar{A} = \begin{bmatrix} -2i & -i \\ 1 & 1-i \end{bmatrix}$, $\bar{B} = \begin{bmatrix} 0 & 1-2i \\ 2i & 3 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & -4+5i \\ 2-2i & 4+5i \end{bmatrix}$, $\overline{AB} = \begin{bmatrix} 2 & -4-5i \\ 2+2i & 4-5i \end{bmatrix}$

$=$

$\bar{A}\bar{B} = \begin{bmatrix} 2 & -4-5i \\ 2+i & 4-5i \end{bmatrix}$