Chapter 1: System of linear equation & Matrices.

1.1 Introduction

1.2 Gaus Elemination

1.3 Matrices

1.1 Introduction

In 2-D a line can be rpresented $ax+by=c$ $a,b\neq 0$

In 3-D a plane " " " $ax+by+cz=d$ $a,b,c\neq 0$

There are 3 ways to solve the system linear eq:

(i) Graphically

(ii) Algbrically using add & sub.

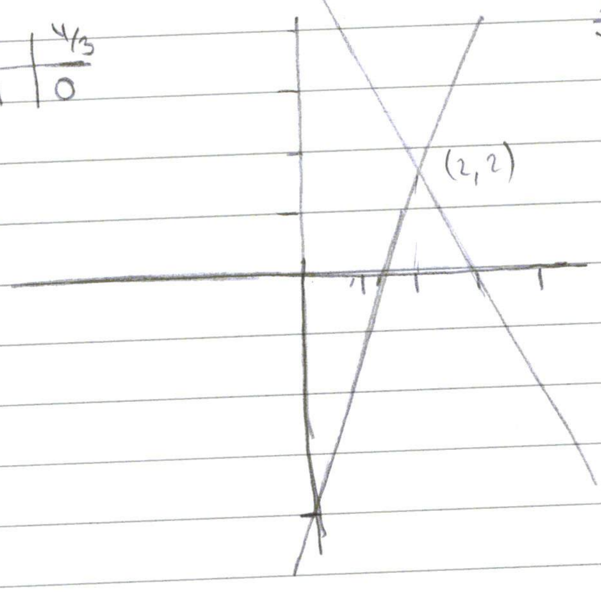
(iii) Using Substitution.

(i) Graphically: Let we have two lines:

$y=3x-4$ & $y=2x+6$

x	0	$4/3$
y	-4	0

x	0	3
y	6	0



b) Solve 1 & 2 Algebraically

we can write eqn 1 & 2 as

$$\begin{aligned} 3x - \cancel{y} &= 4 \\ 2x + \cancel{y} &= 6 \\ \hline 5x &= 10 \end{aligned}$$

$$\boxed{x = 2}$$

$$y = 3x - 4$$

$$= 6 - 4 = 2$$

$$\boxed{y = 2}$$

Systems of Equations

If there are two straight lines L_1 & L_2, Then

1. If L_1 & L_2 intersect exactly at one point Then there is a unique solution.
2. If L_1 & L_2 are coincedent Then are infinity many solution.
3. If L_1 & L_2 are parallel Then there is No-solution.

ex. solve the linear eq.

$$\begin{aligned} x - \cancel{y} &= 1 \quad (1) \\ 2x + \cancel{y} &= 6 \quad (2) \\ \hline 3x &= 7 \end{aligned}$$

$$\boxed{x = \frac{7}{3}}$$

$$\boxed{y = \frac{4}{3}}$$

There is a unique solution

$$\left(\frac{7}{3}, \frac{4}{3}\right)$$

ex. solve the linear eq. $x + y = 4$
$3x + 3y = 6$

The given system of eq. has no solution

* Gauss Elimination

Q. Use Gauss Elimination to solve the system of linear eq.

$$x_1 + 5x_2 = 7$$
$$-2x_1 - 7x_2 = -5$$

Soln. write the aurgamented Matrix of equ 1 & 2

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ -2 & -7 & -5 \end{array}\right]$$

Now using Elementatary Row Transformtions convert it into Row - Echelon form.

$$\begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix} R_2 \to R_2 + 2R_1$$

$$\begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{bmatrix} R_2 \to \frac{R_2}{3}$$

$7 - 5(3)$

$7 - 15$

$$\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix} R_1 \to R_1 - 5R_2$$

The's is The Row - Echeton form

$x_1 = -8$ & $x_2 = 3$

Q_2 Solve the system of linear equation.

$$x - 3y + z = 4$$
$$2x - 8y + 8z = -2$$
$$-6x + 3y - 15z = 9.$$

$$\begin{bmatrix} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ -6 & 3 & -15 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ 0 & -15 & -12 & 21 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & -15 & -12 & 21 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 18 & -54 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\begin{array}{r} 75 \\ -21 \\ \hline 54 \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -8 & 19 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\left[\begin{array}{ccc:c} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & 0 & -1 & 2 \end{array}\right]$$ Check

Now using Back substitution, we can find the value of x, y, z

here $-z = 2$

$\boxed{z = -2}$

&

$-y + 3z = -5$
$-y = -5 - 3z$
$y = 5 + 3z$
$y = 5 + (-6)$
$\boxed{y = -1}$

$x - 3y + z = 4$
$x = 4 - z + 3y$
$x = 4 + 2 - 3$
$= 6 - 3$
$\boxed{x = 3}$

Matrix Properties:

(a) $A+B = B+A$ commulative Low for Addition

(b) $A+(B+C) = (A+B)+C$ Associative Low for addition

(c) $A(BC) = (AB)C$ Associative Low for multiblication

Note: In matrix $AB \neq B\cancel{A}$

ex. consider $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \neq \quad BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

$$AB \neq BA$$

Zero Matrix is denoted by 0 for ex

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2} \quad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}_{1\times 3} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times 1}$$

Identity Matrix an $n\times n$ matrix with ones on the main diagonal & zero else.wher

ex. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Invertible (no-simgular): If $AB = BA = I$, then A is invertible

ex.: $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Theorem 1.4.5 Inverse

In the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad-bc \neq 0$

Then the formula is

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Q: Find the inverse

(a) $A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

sln (a) $(6)(2) - (5)(1) = 12 - 5 = 7$

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 2/7 & -1/7 \\ -5/7 & 6/7 \end{bmatrix}$$

(b) $(-1)(-6) - (3)(2) = 6 - 6 = 0$

Using Row operation find the inverse

$$A = \begin{bmatrix} 1 & 3 \\ -1 & -7 \end{bmatrix}$$

Sln. We can write $[A|I] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -1 & -7 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 + 2R_1$ $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right]$

R x-ve $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & -1 \end{array}\right]$

$R_1 \to R_1 - 3R_2$ $\left[\begin{array}{cc|cc} 1 & 0 & 7 & 3 \\ 0 & 1 & -2 & -1 \end{array}\right]$

$$A^{-1} = \begin{bmatrix} 7 & 3 \\ -2 & -1 \end{bmatrix}$$

__Ex.__ Using Row operation find A^{-1}

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$ $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$

$R_3 \to R_3 + 2R_2$ $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$

$R_3 \times -ve$ $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 1 \\ 0 & 0 & 1 & 5 & -2 & 1 \end{array}\right]$

$R_1 \to R_1 - 3R_3$
$R_2 \to R_2 + 3R_3$ $\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 1 & 0 & 5 & -2 & -1 \end{array}\right]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 19 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Theorem 1.6.2

if A is invertible $n \times n$ matrix, then for each $n+1$ matrix b, the system for each $n+1$ matrix b, The system of equation $A\cancel{X}\cancel{B} = b$ has exactly one solution $X = A^{-1} b$

ex. Find the solution of system of linear eq. using A^{-1}

$$x + 3y = 1$$
$$2x + 5y = 3$$

we can writ $[A \mid I] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right]$

$R_2 \Rightarrow R_2 - 2R_1$ $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$

$R_2 * -ve$ $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]$ $\quad R_1 \to R_1 - 3R_2$ $\left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right]$

$A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$, $\quad X = A^{-1} b = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$\boxed{x = 4, \ y = -1}$

Chapter 2

Determinants

<u>Minors & cofactor:</u> if A is a square matrix the minor of a_{ij} is denoted by M_{ij}. The number $(-1)^{i+j} M_{ij}$ is denoted C_{ij} is called the cofactor of a_{ij}.

<u>Ex.</u> find the minors & cofactor of

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

<u>Soln.</u> the minor of a_{11} is $M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$

the cofactor of a_{11} is

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (16) = 16$$

the minor of a_{12} is $M_{12} = \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = 10$

the cofactor of a_{12} is

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 (10) = -10$$

simillary the miror of a_{13} is $M_{13} = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3$

and the cofactor $C_{13} = (-1)^{1+3} M_{13} = (-1)^4 (3) = 3$

$M_{32} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$ $\quad C_{32} = (-1)^{2+3} M_{23} = (-1)^5 (26) = -26$

~~cofactor~~

Cofactor expansion Along Row wise:

Ex. Find the determentant by cofactor expansion along first Row then first column.

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

Soln. (a) $\det(A) = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$

$= 3(-4) - 1(-11) + 0 = -12 + 11 = -1$

(b) $\det(A) = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$

$= 3(-4) - (-2)(-2) + 5(3) = -12 - 4 + 15$

$= -1$

Technique to evaluate 2x2 and 3x3 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right]$$

$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$
$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

Ex. Calculate $A = \left|\begin{array}{ccc|cc} 1 & 5 & -3 & 1 & 5 \\ 1 & 0 & 2 & 1 & 0 \\ 3 & -1 & 2 & 3 & -1 \end{array}\right|$

$$= (1)(0)(2) + (5)(2)(3) + (-3)(1)(-1) - (-3)(0)(3) - (1)(2)(-1) - (5)(1)(2)$$

$$= 0 + 30 + 3 + 0 + 2 - 10 = 25$$

<u>Theorem 2.2.5</u> If A is a square matrix with two proponal Rows or two propotional columns then
$\det(A) = 0$ ~~fo~~

<u>ex.</u>

$$\begin{vmatrix} 1 & 3 & -2 & 4 \\ 1 & 6 & -4 & 8 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\begin{vmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix}$$

Q. Evalua~~te~~ <u>the determinate by Row Reduction</u>.

In this we convert the determinant into upper triangular matrix using Row operation.

Q. Evaluate $\operatorname{Det}(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix}$

$\boxed{R_2 \leftrightarrow R_1}$ $A = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$

take common 3 $= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$

$R_3 \to R_3 - 2R_1$ $= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$

$R_3 \to R_3 - 10R_2 = -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$

$\cancel{R_3 \to R}$

$= (-3)(-55) \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$

$= (-3)(-55)(1)$

$= 165$

Theorem 2.3.3

A square matrix A is invertable iff $\det A \neq 0$

Theorem: if A & B a square matrix of the same size then $\det(AB) = \det(A) \cdot \det(B)$

Cramers Rule:

If $Ax = b$ is a system of n-linear equation is n unknowns such that $\det(A) \neq 0$, then the system has a unique solution, this solutions is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)} \cdots \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where $A_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{23} & a_{33} \end{vmatrix}$ $\quad A_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$

$A_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$

Q: Using cramers Rule to solve

$$x_1 + \quad + 2x_3 = 6$$
$$-3x_1 + 4x_1 + 6x_3 = 30$$
$$-x_1 - 2x_1 + 3x_3 = 8$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

$\det(A) = 44$, $\det(A_1) = -40$

$\det(A_2) = 72$, $\det(A_3) = 152$

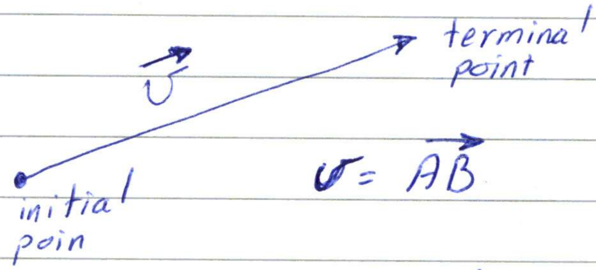
$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11} = \frac{-10}{11}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

Chapter 3 Vectors

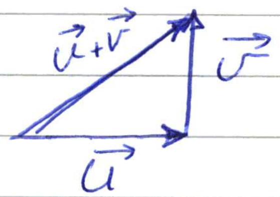
Vector can be represent in 2-D or 3-D



$v = \overrightarrow{AB}$

zero vector is denoted by $\vec{0}$, in 2-D $\vec{0} = (0,0)$ and 3-D $\vec{0} = (0,0,0)$

Addition:



component of the vector:

if we have 2 vectors $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, Then component of vectors is given by

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$$

Q. Find the component of a vector with initial point $P_1(2,-1,4)$ & terminal point $P_2(7,5,-8)$

$$\overrightarrow{P_1P_2} = (7-2,\ 5-(-1),\ (-8)-4) = (5, 6, -12)$$

Q: if $\vec{v} = (1,-3,2)$ & $\vec{w} = (4,2,1)$
find $\vec{v}+\vec{w}$ & $\vec{v}-\vec{w}$

$\vec{v}+\vec{w} = (1+4,\ -3+2,\ 2+1) = (5,-1,3)$

$\vec{v}-\vec{w} = (1-4,\ -3-2,\ 2-1) = (-3,-5,1)$

Theorem 3.1 If $\vec{v}, \vec{u}, \vec{w}$ are the vectors in R^n & if k & m are scalers, then

(a) $\vec{u} + \vec{v} =$

(b) $(\vec{u}+\vec{v})+\vec{w} = \vec{u} + (\vec{v}+\vec{w})$

(c) $\vec{u} + 0 = 0 + \vec{u} = \vec{u}$

(d) $k(\vec{u}+\vec{w}) = k\vec{u} + k\vec{w}$

(e) $0 \cdot \vec{v} = 0$

(f) $k \cdot \vec{0} = 0$

(g) $1 \cdot \vec{u} = \vec{u}$

Norm of vector

The length of a vector is defined by $\|\vec{v}\|$ which is called norm of $\vec{v}$ or length of $\vec{v}$

in R^2 $\|\vec{v}\| = \sqrt{\vec{v}_1^2 + \vec{v}_2^2}$

in R^3 $\|\vec{v}\| = \sqrt{\vec{v}_1^2 + \vec{v}_2^2 + \vec{v}_3^2}$

Q: Find the norm of a vector $v = (-3, 2, 1)$ in R^3

$$\|\vec{v}\| = \sqrt{(-3)^2 + (2)^2 + (1)^2} = \sqrt{9+4+1} = \sqrt{14}$$

Unit vector $\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v}$

standard unit vector in R^2 or in R^3

in R^2 $i = (1,0)$ & $j (0,1)$

in R^3 $i = (1,0,0)$, $j (0,1,0)$ & $k = (0,0,1)$

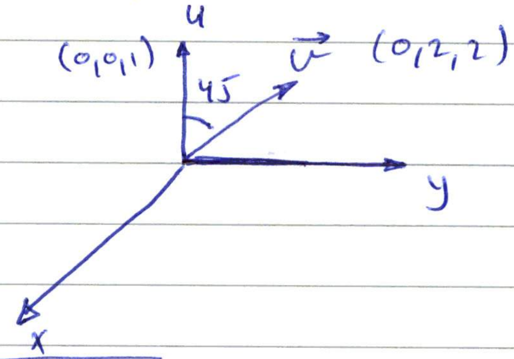
Dot Product: If θ is the angle between $\vec{u}$ & $\vec{v}$, Then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

or

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

Q: Find the dot product of figure:



$\|\vec{u}\| = \sqrt{(0)^2 + (0)^2 + (1)^2} = 1$

$\|\vec{v}\| = \sqrt{(0)^2 + (2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2}$

$\cos\theta = \cos 45 = \frac{1}{\sqrt{2}}$

Now $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta = (1)(2\cancel{\sqrt{2}}) \frac{1}{\cancel{\sqrt{2}}} = 2$

Q: Find the dot product of $a = (1,2,3)$ & $b = (4,-5,6)$

$\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = (1)(4) + (2)(-5) + (3)(6)$

$= 4 - 10 + 18 = 12$

Q: if $a = (6,-1,3)$ for what value of c is the vector $b = (4, c, -2)$ perpendicular to a

$a \cdot b = 24 + (-1)c - 6 = 18 - c$

perpendicular $\longrightarrow \cos 90 = 0$

$c = 18$ $\longleftarrow$ $a \cdot b = 0 = 18 - c$

Orthogonality: Two vectors are orthogonal iff $\vec{u} \cdot \vec{v} = 0$

Q: show that $\vec{u} = (-2, 3, 1, 4)$ and $\vec{v} = (1, 2, 0, -1)$ are orthogonal. in R^4

Theorem if a & b are constant & non-zero Then equation of the form

$$ax + by + c = 0$$

represent a line in R^2 with normal a & b.

Theorem 3.3.4

In R^2 the distance between the point $P_0(x_0, y_0)$ and the line $ax + by + c = 0$

is $$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

similary in R^3

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Q: Find the distance between the point $(1,-4,-3)$ and the plane $2x-3y+6z=-1$

$$2x-3y+6z+1=0$$

$$D=\frac{|(2)(1)+(-3)(-4)+(6)(-3)+1|}{\sqrt{(2)^2+(-3)^2+(6)^2}}=\frac{|2+12-18+1|}{7}$$

$$=\frac{|-3|}{7}=\frac{3}{7}$$

Q: Two planes are given $x+2y-2z=3$ & $2x+4y-4z=7$ Find the distance between them.

~~put~~ or we can write

$$x+2y-2z-3=0 \quad —(1)$$
$$2x+4y-4z-7=0 \quad —(2)$$

put in (1) $y=z=0 \rightarrow x=3$

point is $(3,0,0)$

$$D=\frac{|(3)(2)+(4)(0)+(-4)(0)+(-7)|}{\sqrt{(2)^2+(4)^2+(-4)^2}}=\frac{|6-7|}{6}=\frac{|-1|}{6}$$

$$=\frac{1}{6}$$

Defination If x_0 & $\vec{v}$ are vectors in R^n and if $\vec{v}$ is non-zero,

Then : the equation $x = x_0 + tv$ defines a line through x_0. That is parallel to v

similary R^3

$$x = x_0 + t_1 u_1 + t_2 u_2$$

Q: Find ~~of~~ a vector equation & parametric equation of line in R^3 that pass throug the point $P_0(1, 2, -3)$ & is paralle the vector $u = (4, -5, 1)$

we know the eq. of the line $x = x_0 + t\vec{u}$

here $x_0 = (1, 2, -3)$ $u = (4, -5, 1)$

$$(x, y, z) = (1, 2, -3) + t(4, -5, 1)$$

$x = 1 + 4t$

$y = 2 - 5t$

$z = -3 + t$

3.5 Cross Product

if $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$

cros product $\vec{u} \times \vec{v} =$

$= (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$

$$\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \Rightarrow u \times v$$

$$\Rightarrow \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

Q: Calculate the cross prodct

$\vec{u} = (1, 2, -2)$, $\vec{v} = (3, 0, 1)$ $\begin{bmatrix} 1 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix}$

$$\vec{u} \times \vec{v} = \left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \right)$$

$= \cancel{(2, -(6)}$

Chap 4 Vector Spaces

Row Space, column space & Null space

Let the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & & \cdots & a_{mn} \end{bmatrix}$

The vectors $r_1 = [a_{11}, a_{12}, a_{13}, \cdots, a_{1n}]$

$r_2 = [a_{21}, a_{22}, \cdots\cdots, a_{2n}]$

$\vdots$

r_m

are called the Row vector

and the vectors

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \cdots\cdots$$

are called the column vector.

Q: Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$

$r_1 = [2 \quad 1 \quad 0]$, $r_2 \; [3 \quad -1 \quad 4]$

$c_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $C_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

Row Space The subspace of R^n spanned by Row vectors of A is called the Row space.

Column Space //

Null space: The solution space of $Ax = 0$ where is the subspace of R^n is called the null space of A.

Q: Find the basis for Row and column space

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ 1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Soln. Reduced to Row echelon form

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The basis vector are (non zero Row)

$r_1 = [1 \quad -3 \quad 4 \quad -2 \quad 5 \quad 4]$

$r_2 = [0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6]$

$r_3 = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5]$

Now we observe that first, third, fifth column contains the leading 1. so

$$C_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_3' = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_5' = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad C_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

Q: The matrix $R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

<u>Soln.</u> this is already in Row echelon form

$$r_1 = [1 \quad -2 \quad 5 \quad 0 \quad 3]$$

$$r_2 = [0 \quad 1 \quad 3 \quad 0 \quad 0]$$

$$r_3 = [0 \quad 0 \quad 0 \quad 1 \quad 0]$$

& $C_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Q: Find the basis for the Row space:

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Soln.

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}$$

convert it into Row echelon form.

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$c_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_2' = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_4' = \begin{bmatrix} 2 \\ -10 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ -3 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{bmatrix}, \quad c_4 = \begin{bmatrix} 2 \\ 6 \\ 18 \\ 8 \\ 6 \end{bmatrix}$$

$r_1 = [1 \quad -2 \quad 0 \quad 0 \quad 3]$

$r_2 = [2 \quad -5 \quad -3 \quad -2 \quad 6]$

$r_4 = [2 \quad 6 \quad 18 \quad 8 \quad 6]$

Rank & Nullity

The common dimension of the row space & column space of the matrix A is called the Rank of A.
The dimension of the null space of A is called the nullity (A)

Q: Find the rank & nullity of matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Soln. Using Row echelon form

$$A = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

the rank of A is 2 because non zero is 2

(No. of non-zerou Row)

Now we have 4 equation 6 unknown

Dimension Theorem :

$$\text{Rank}(A) + \text{nullity}(A) = n \quad \text{number of column}$$

we know, by Rank Nullity theorem

$$\text{Rank}(A) + \text{Nullity}(A) = n$$

$$2 + \text{Nullity}(A) = 6$$

$$\text{Nullity}(A) = 6 - 2$$

$$= 4$$

Q: let $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix}$

a) Find the Rank & Nallity of A

b) Find ~~a~~ subset of the column vectors of that form a basis for the column space of A a)

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(A) = 3$

$\text{Nullity}(A) = n - \text{Rank}(A) = 5 - 3 = 2$

b)

$$C_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_2' = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_4' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Chapter 5

Eigen Value & Eigen Vector

Defn:

Let A be a square matrix, and λ be any value.

Then $Ax = \lambda x \rightarrow (1)$

has a ~~Non solution~~ non-trivial solution are called Eigen Value of A.

Corresponding to Eign value λ, $\exists$ a non-zero vector x, such that

$|\lambda I - A| x = 0$, Then x is called The Eign Vector

Remark: (1) $\boxed{|\lambda I - A| = 0}$ is called th characteristi equation.

(2) The Eign value of triangular matrix are its diagonal elements

Ex. Determine the Eign Value and the Eign Vectors of $\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$

step 1: The characteristic equation of A is given by $|\lambda I - A| = 0$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -1 \\ -6 & \lambda - 2 \end{vmatrix} = 0$$

$$= (\lambda - 3)(\lambda - 2) - 6 = 0$$

$$\lambda^2 - 5\lambda + 6 - 6 = 0$$

$$\lambda(\lambda - 5) = 0$$

$$\boxed{\lambda = 0, 5}$$

$\boxed{\lambda_1 = 0, \lambda_2 = 5 \text{ are Eign Values of } A.}$

Step 2: To find the Eign vectors:

(a) corresponding $\lambda_1 = 0$

$$(\lambda_1 I - A)X = 0$$

$$\begin{pmatrix} 0-3 & -1 \\ -6 & 0-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -3 & -1 \\ -6 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$-3x_1 - x_2 = 0 \longrightarrow ①$$

$$-6x_1 - 2x_2 = 0 \longrightarrow ②$$

since the rank of matrix is one.

so we choose eqn ①

$$-x_2 = 3x_1$$

$$x_2 = -3x_1$$

or we can write

$$= \{(x_1, -3x_1) \in \forall x_1 \text{ is scaller}\}$$

$$= \{x_1(1, -3) \in \forall x_1 \text{ is scaller}\}$$

$(1, -3)$ are the Eign vector

(b) Eign vector corresponding $\lambda_2 = 5$

$$(\lambda_2 I - A)X = 0$$

$$\begin{pmatrix} 5-3 & -1 \\ -6 & 5-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 & -1 \\ -6 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$2x_1 - x_2 = 0 \rightarrow (1)$
$-6x_1 + 3x_2 = 0 \rightarrow (2)$

since the Rank of the matrix is one. so
so we choose eqz (1)

$$2x_1 = x_2$$

$= \{ (x_1, 2x_1) \in \forall \ x_1 \text{ is scaller} \}$

$= \{ x_1(1, 2) \in \forall \ x_1 \text{ is scaller} \}$

$(1, 2)$ are the Eigen vector vectors.

Ex. Find the Eign value of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

The characteristic equation is:

$$|\lambda I - A| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} = 0$$

$$\lambda - (-1)$$

$$\underbrace{(\lambda - 3)(\lambda + 1)} + 8 = 0$$

$$\cancel{\lambda^2 - 3\lambda + \lambda - 3 + 8} = 0$$

$$\cancel{\lambda^2 - 2\lambda + 5} = 0$$

$\lambda = 3, -1 \rightarrow$ the eign value of A

Ex. Find the Eign value of $A = \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 2/3 & 0 \\ 5 & -8 & -1/4 \end{bmatrix}$

Since it is a lower triangle matrix so the Eign value its diagonal ~~m~~element

$$\lambda_1 = 1/2 \quad , \quad \lambda_2 = 2/3 \quad , \quad \lambda_3 = -1/4$$

Ex. Find the Eign value of $A \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$

the characteristic eqn.

$|\lambda I - A| = 0$

$\begin{vmatrix} \lambda+2 & +1 \\ -5 & \lambda-2 \end{vmatrix} = 0 \Rightarrow (\lambda+2)(\lambda-2)+5=0$

$\lambda^2+2\lambda\,\cancel{+}\,2\lambda-4+5=0$

$\lambda^2+1=0$

$\lambda^2+1=0$

$\lambda = \pm i$

Diagonalization :

Step 1: check The E.V. are L.I.
step 2: Form the matrix $P = [P_1, P_2, \dots]$
step 3: The matrix $P^{-1}AP$ will be diagonal.

Ex. Find the E.V & E.vector, basis & diagonalise the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

step 1 $|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda & 0 & +2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$
$$(\lambda-1)(\lambda-2)^2 = 0$$

$$\lambda = 1, 2, 2$$

Step 2 E.V corresponding to $\lambda_1 = 1$

$$(\lambda_1 I - A)X = 0$$

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

on solving

$x_1 = -2s$, $x_2 = s$, $x_3 = s$

$(-1, 1, 1) \Rightarrow$ the eign vector

E.V. corresponding $\lambda_2 = 2$

$$(\lambda_2 I - A)\cancel{X} = 0$$

$$\begin{pmatrix} 2 & 0 & +2 \\ -1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$(-1, 0, 1)$

EV. corresponding $\lambda_3 = 2$

$(0, 1, 0)$

$$P_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \; P_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; P_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagondise A, we have to write $P^{-1}AP$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Chapter 6

Inner Product Space

Defn: Let u, v, w be a vectors in a vector space V, and C any constant.
Then inner product $\langle u, v \rangle$ if it satisfy the following axioms.

Axiom 1: $\langle u, v \rangle = \langle v, u \rangle$

Axiom 2: $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

Axiom 3: $C\langle u, v \rangle = \langle Cu, v \rangle$

Axiom 4: $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$

Note: A vector space V is $(V, +, \cdot)$ with an inner product is called inner product space $\langle V, +, \cdot, \langle , \rangle \rangle$)

Q: Let $u = (u_1, u_2)$ and $V = (v_1, v_2)$ be the vectors in R^2
Varify that the Euclidean Inner Product
$\langle u, v \rangle = 3u_1 v_1 + 2u_2 v_2$
satisfy the four Inner product Axioms.

Soln.

Axiom 1: $\langle u, v \rangle = 3u_1 v_1 + 2u_2 v_2$
$= 3v_1 u_1 + 2v_2 u_2 = \langle v, u \rangle$

Axiom 2: $\langle u, v+w \rangle = 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2$

$= 3(u_1 w_1 + v_1 w_1) + 2(u_2 w_2 + v_2 w_2)$

$= 3u_1 w_1 + 3v_1 w_1 + 2u_2 w_2 + 2v_2 w_2$

$= (3u_1 w_1 + 2u_2 w_2) + (3v_1 w_1 + 2v_2 w_2)$

$= \langle u, w \rangle + \langle v, w \rangle$

Axiom 3: $\langle cu, v \rangle = 3(cu_1)v_1 + 2(cu_2)v_2$

$= c(3u_1 v_1 + 2u_2 v_2)$

$= c\langle u, v \rangle$

Axiom 4: $\langle u, v \rangle = 3(v_1, v_1) + 2(v_2, v_2)$

$= 3v_1^2 + 2v_2^2 \geqslant 0$

with equality iff $v_1 = v_2 = 0$

Q: show that the function defined inner produt in $\mathbb{R}^2$ where $u=(u_1, u_2)$ & $v=(v_1, v_2)$

$$\langle u, v \rangle = u_1 v_1 + 2u_2 v_2$$

Axiom 1: $\langle u, v \rangle = u_1 v_1 + 2u_2 v_2$

$= v_1 u_1 + 2v_2 u_2$

$= \langle v, u \rangle$

Axiom 2: $\langle u, v+w \rangle = u_1(v_1 + w_1) + 2u_2(v_2 + w_2)$

$= u_1 v_1 + u_1 w_1 + 2u_2 v_2 + 2u_2 w_2$

$= u_1 v_1 + 2u_2 v_2 + u_1 w_1 + 2u_2 w_2$

$= \langle u, v \rangle + \langle u, w \rangle$

Axiom 3: $c\langle u, v \rangle = c(u_1 v_1 + 2u_2 v_2)$

$= (cu_1)v_1 + 2(cu_2) v_2$

$= \langle cu, v \rangle$

Axiom 4: $\langle v, v \rangle = v_1 v_1 + 2 v_2 v_2$

$= v_1^2 + 2v_2^2$

if $\langle v, v \rangle = 0 \Rightarrow v_1^2 + v_2^2 = 0$

$\Rightarrow v_1 = v_2 = 0$

Properties of inner product

① $\langle 0, v \rangle = \langle v, 0 \rangle = 0$

② $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

③ $\langle u, cv \rangle = c\langle u, v \rangle$

④ $\|u\| = \sqrt{\langle u, u \rangle}$

⑤ Distance between u and v

$$d(u,v) = \|u-v\| = \sqrt{\langle u-v, u-v \rangle}$$

⑥ Angle between two non-zero vectors u and v

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\| \, \|v\|} \qquad 0 \leqslant \theta \leqslant \pi$$

Note: if $u \perp v$ then $\langle u, v \rangle = 0$ orthogonal

if $\|u\| = 1$ then it is called a unit vector

Q. Calculating the inner product
$\langle u-2v, 3u+4v\rangle$

$= \langle u, 3u+4v\rangle - \langle 2v, 3u+4v\rangle$

$= \langle u, 3u\rangle + \langle u, 4v\rangle - \langle 2v, 3u\rangle - \langle 2v, 4v\rangle$

$= 3\langle u, u\rangle + 4\langle u, v\rangle - 6\langle v, u\rangle - 8\langle v, v\rangle$

$= 3\|u\|^2 + 4\langle u, v\rangle - 6\langle v, u\rangle - 8\|v\|^2$

$= 3\|u\|^2 - 2\langle u, v\rangle - 8\|v\|^2$

Q: Show that the following set is an ~~orthogonal~~ orthonormal basis:

$$S = \{\underbrace{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}_{v_1}, \underbrace{\left(\frac{-\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)}_{v_2}, \underbrace{\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)}_{v_3}$$

$$v_1 \cdot v_2 = \frac{1}{\sqrt{2}}\left(\frac{-\sqrt{2}}{6}\right) + \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{6}\right) + 0\left(\frac{2\sqrt{2}}{3}\right)$$

$$= -\frac{1}{6} + \frac{1}{6} = 0$$

$$v_1 \cdot v_3 = \frac{1}{\sqrt{2}}\left(\frac{2}{3}\right) + \frac{1}{\sqrt{2}}\left(-\frac{2}{3}\right) + 0\left(\frac{1}{3}\right)$$

$$= \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} = 0$$

$$v_2 \cdot v_3 = \frac{-\sqrt{2}}{6}\left(\frac{2}{3}\right) + \frac{\sqrt{2}}{6}\left(\frac{-2}{3}\right) + \frac{2\sqrt{2}}{3}\left(\frac{1}{3}\right)$$

$$= \frac{-\sqrt{2}}{6} - \frac{\sqrt{2}}{6} + \frac{2\sqrt{2}}{9} = 0$$

Now for normal

$$\|v_1\| = \sqrt{v_1 \cdot v_1} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2} = 1$$

$$\|v_2\| \sqrt{v_2 \cdot v_2} = \sqrt{\left(\frac{\sqrt{2}}{6}\right)^2 + \left(\frac{\sqrt{2}}{6}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2} = 1$$

$$\|v_3\| = \sqrt{v_3 \cdot v_3} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = 1$$

So S is an orthonormal.

Q: In $P_3(x)$ with inner product $\langle \cancel{P, v} \rangle$

$\langle P, v \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$

The standard basis $B = \{1, x, x^2\}$

is orthonormal

$v_1 = 1 + 0x + 0x^2 \quad v_2 = 0 + x + 0x^2 \quad v_3 = 0 + 0x + x^2$

$\|v_1\| = \sqrt{v_1 \cdot v_1} = \sqrt{(1)(1) + 0 \cdot 0 + 0 \cdot 0} = 1$

$\|v_2\| = \sqrt{v_2 \cdot v_2} = \sqrt{0 \cdot 0 + (1)(1) + 0 \cdot 0} = 1$

$\|v_3\| = \sqrt{v_3 \cdot v_3} = 1$

Now for orthogonal

$v_1 \cdot v_2 = (1)(0) + (0)(1) + (0)(0) = 0$

$v_1 \cdot v_3 = (1)(0) + (0)(0) + (0)(1) = 0$

$v_2 \cdot v_3 = (0)(0) + (1)(0) + (0)(1) = 0$

So it is an orthonormal.

Week 10

Chapter 7 Diagonalization of Matrix

Orthogonal Matrix: A square matrix is said to be orthogonal if its transpose is same as its inverse ie $A^{-1} = A^T$ or $AA^T = I$

Q: Check whether it is orthogonal $A = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$

S. Transpose of A is $A^T \begin{bmatrix} 3/7 & -6/7 & 2/7 \\ 2/7 & 3/7 & 6/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix}$

Now $$A A^T = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix} \cdot \begin{bmatrix} 3/7 & -6/7 & 2/7 \\ 2/7 & 3/7 & 6/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Orthogonally Diagonalizing of $n \times n$ Symmetric Matrix

Step#1: Find the Eigen Values.

Step#2: Find the Eigen Vectors.

Step#3: Find P where $P = [P_1, P_2, P_3]$ such that $$P^T A P = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

Q: Find an orthogonal Matrix P that diagonalize

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

S: Step #1: $|\lambda I - A| = 0$

$$|\lambda I - A| = \begin{bmatrix} \lambda-4 & -2 & -2 \\ -2 & \lambda-4 & -2 \\ -2 & -2 & \lambda-4 \end{bmatrix}$$

$$= (\lambda-2)^2(\lambda-8) = 0$$

$\lambda = 2, 2, 8$ are the Eigen Values

Step #2: The Eigen Vector corresponding $\lambda = 2, 2$ are

$$v_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 2/\sqrt{2} \end{bmatrix}$$

and corresponding to $\lambda = 8$

$$v_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 1/\sqrt{3} \end{bmatrix}$$

Step #3: Now $P = [P_1, P_2, P_3]$

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$P^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$P^TAP = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Quadratic Forms:

A homogeneous polynomial of second degree in any number of variables is called a quadratic form for ex.

$$a_1 x_1^2 + a_2 x_2^2 + 2a_3 x_1 x_2$$

$$\Rightarrow \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X^T A X$$

similirly

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + 2a_4 x_1 x_2 + 2a_5 x_1 x_3 + 2a_6 x_2 x_3$$

Then

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = X^T A X$$

Q. Express the quadratic form in Matrix Notation

(a) $2x^2 + 6xy - 5y^2$

(b) $x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1 x_2 - 2x_1 x_3 + 8x_2 x_3$

(a) $2x^2 + 6xy - 5y^2 = [x\ y]\begin{bmatrix} 2 & 3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(b) $x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 8x_2x_3$

$$[x_1\ \ x_2\ \ x_3]\begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 4 \\ -1 & 4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Theorem: If A is a symatric matrix, Then

(a) x^TAx is positive definit Eigen value of $A > 0$

(b) X^TAX is negative definit Eign value of $A < 0$

(c) X^TAX is Indefinit iff at least one positive & one negative.

Q: Find the nature of the Quadratic form

(a) $x^2 + 5y^2 + z^2 + 2xy + 2yz + 6zx$

(b) $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$.

(a) $x^2 + 5y^2 + z^2 + 2xy + 2yz + 6zx$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$, The Eigen values of A is $-2, 3, 6$

so give quadratic form is Indefinit from part c of theorrem

(b) $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The Eigen values of A are $2, 3, 6$

so the given quadratic form is positive defint

Conjugate Transpose:

If A is complex, Then $A^* = \bar{A}^T$

Q: Find a conjugate transpose of $A = \begin{bmatrix} 1+i & -i & 0 \\ 2 & 3-2i & i \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 1-i & i & 0 \\ 2 & 3+2i & -i \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} 1-i & 2 \\ i & 3+2i \\ 0 & -i \end{bmatrix} = A^*$$

Defn. (a) unatry if $A^{-1} = A^*$

Hermitian if $A^* = A$

(b) E.V of H.M is real

Q. Find the E.V of Hermation matrix

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$$

S. The characterist equation is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1-i \\ -1+i & \lambda - 3 \end{vmatrix}$$

$$= (\lambda-2)(\lambda-3) - (-1-i)(-1+i)$$

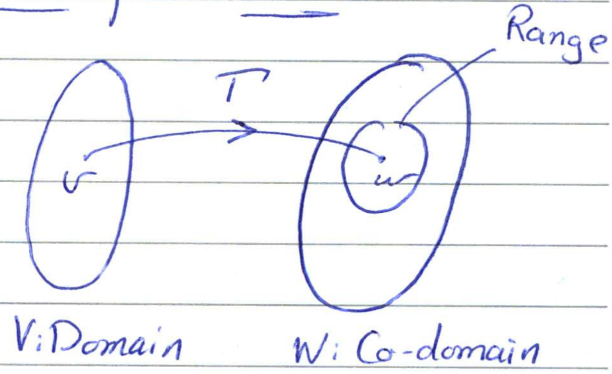
$$= (\lambda^2 - 5\lambda + 6) - 2$$

$$= (\lambda-1)(\lambda-4)$$

$\lambda = 1, 4$ which ir real.

Week 11

Chapter 8: Linear Transformation.



Function T that maps a vector space V into a vector space w.

Q: $T: R^2 \longrightarrow R^2$ $v = (v_1, v_2) \in R^2$, $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$

(a) Find the image of $v(-1, 2)$ (b) Find the Pre-image of $w = (-1, 11)$

Soln.

(a) $v = (-1, 2)$

$T(v) = T(-1, 2) = (-1-2, -1+2(2)) = (-3, 3)$

(b) $T(v) = w = (-1, 11)$

we know $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$

so compare the elements

$v_1 - v_2 = -1$

$-v_1 \pm 2v_2 = -11$

$-3v_2 = -12$

$v_2 = 4$

$v_1 - 4_2 = -1$

$v_1 = -1 + v_2 = -1 + 4 = 3$

$v_1 = 3$ $v_2 = 4$

$(3, 4)$ is the image of $(-1, 11)$

Linear Transformation: Let V, W be the V.S

$T: V \to W$: V to W Linear Transformation (L.T).

if (a) $T(u+v) = T(u) + T(v) \quad \forall u, v \in V$

(b) $T(cu) = cT(u) \quad \forall c \in R$

Q: Verify a L.T. from R^2 into R^2

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) \rightarrow 1$$

Soln: Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ & c is any real nomber.

(a) Let $u + v = (u_1 + u_2) + (v_1 + v_2)$

$= (u_1 + v_1, u_2 + v_2)$

so $T(u+v) = T(u_1 + v_1, u_2 + 2v_2)$

$= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2))$

$= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2))$

$= ((u_1 - u_2), (u_1 + 2u_2) + ((v_1 - v_2), (v_1 + 2v_2))$

$= T(u) + T(v)$ from 1

(b) let $cu = c(u_1, u_2) = (cu_1, cu_2)$

$T(cu) = T(cu_1, cu_2)$

$= ((cu_1 - cu_2), (cu_1 + 2cu_2))$

$= (c(u_1 - u_2), c(u_1 + 2u_2))$

$= c((u_1 - u_2), (u_1 + 2u_2))$

$= cT(u)$

Therefore T is a Linear Transformation

Zero Transformation: $T: V \to W$, $T(v) = 0 \quad \forall v \in W$

Idetity Transformation: $T: V \to W$, $T(v) = v \quad \forall v \in V$

Q: Let $T: R^3 \to R^3$ be a L.T such that

$$T(1,0,0) = (2,-1,4)$$
$$T(0,1,0) = (1,5,-2)$$
$$T(0,0,1) = (0,3,1)$$

Find $T(2,3,-2)$

Soln. we can write

$$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$$

$$T(2,3,-2) = 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1)$$
$$= 2(2,-1,4) + 3(1,5,-2) - 2(0,3,1)$$
$$= (4,-2,8) + (3,15,-6) - (0,6,2)$$
$$= (7,7,0)$$

Q: The function $T: R^2 \to R^3$ is defined by

$$T(v) = Av = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Find $T(v)$ where $v(2,-1)$

Soln. $v = (2,-1)$

$$T(v) = Av = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

$$T(2,-1) = (6,3,0)$$

Kernal of L.T:

Let $T: V \to W$ be a L.T Then the set of all vectors v in V that satisfy $T(v)=0$ is called the kernal of T. and it is denoted by ~~ker(T)~~

$$\ker(T) = \{ v = T(v) = 0 \ \forall v \in V \}$$

or Null space of T.

Range of L.T:

Let $T: V \to W$ be a L.T.

Then the set of all vectors ~~w~~ in W That are images of vectors in V is called range of T and it is denoted by Range

Rank of L.T.

rank(T) = The dimension of The range of T

Nullity of L.T:

nullity (T) : The dimension of kernal of T

Dimension Theorem for a L.T:

Let $T: V \to W$ be a L.T

$$\text{rank}(T) + \text{nullity}(T) = n$$

OR

$$\dim(\text{range of } T) + \dim(\ker T) = \dim(\text{domain of } T)$$

Q: Find the rank and nullity of a L.T

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Soln. Rank $= 2$ No. of Non-zero Rows.

and $n = 3$

so by Rank nullity Theorem

$\text{rank}(T) + \text{nullity}(T) = n$

nullity ~~rank~~ $(T) = 3 - 2 = 1$

one-one:

onto:

Q: Find the standard matrix for the L.T
$T: R^3 - R^2$ defined by
$T(x,y,z) = (x-2y, 2x+y) \longrightarrow ①$

$T(e_1) = T(1,0,0) = (1-2\times 0, 2\times 1+0)$
$= (1,2)$ from ①

In Matrix Notation

$$T(e_1) = T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

$T(e_2) = T(0,1,0) = (0-2\times 1, 2\times 0+1) = (-2,1)$

$$T(e_2) = T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}$$

$T(e_3) = T(0,0,0) = (0,0)$

$$T(e_3) = T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$A = [T(e_1), T(e_2), T(e_3)] = \begin{bmatrix}1 & -2 & 0\\2 & 1 & 0\end{bmatrix}$$

Week 12: Chapter 9 Numerical Methods

LU-Decomposition:

Step 1: Reduce the matrix to Row Echelon form.

Step 2: In each position along the main diagonal of L, place Reciprocal of the multiplier.

Step 3: In each position along the main diagonal of L, place ~~Place~~ the negative of the multiplier.

Step 4: From the decomposition $A = LU$

Q: Find a LU-decompositon $A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$

Soln. Given

$$U = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \qquad L = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}$$

multipler $\frac{1}{6}$ $= \begin{bmatrix} 1 & -1/3 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$

multipler -9
multipler -3 $= \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 2 & 1 \\ 0 & 8 & 5 \end{bmatrix}$

multipler $\frac{1}{2}$ $= \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 8 & 5 \end{bmatrix}$

multipler -8 $= \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$

multipler -1 $= \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$

$$A = LU = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

The Power Method:

step1: choose an arbitrary non-zero vector.

step2: Compute Ax_0 and multiply it by $\frac{1}{max(Ax_0)}$

step3: Compute Ax_1 and multiply it by $\frac{1}{max(Ax_1)}$

⋮

Q: Apply the power method with maximum entry scaling to $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ with $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ upto x_5.

Soln. $Ax_0 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

and $x_1 = \frac{Ax_0}{max(Ax_0)} = \frac{1}{3} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cong \begin{bmatrix} 1.000 \\ 0.6667 \end{bmatrix}$

Now
$$Ax_1 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.667 \end{bmatrix} \cong \begin{bmatrix} 4.3333 \\ 4.0000 \end{bmatrix}$$

$$x_2 = \frac{Ax_1}{max(Ax_1)} = \frac{1}{4.333} \begin{bmatrix} 4.333 \\ 4.000 \end{bmatrix} \cong \begin{bmatrix} 1.00000 \\ 0.92308 \end{bmatrix}$$

$$Ax_2 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1.00000 \\ 0.92308 \end{bmatrix} \cong \begin{bmatrix} 4.84615 \\ 4.76923 \end{bmatrix}$$

$$x_3 = \frac{Ax_2}{max(Ax_2)} = \frac{1}{4.84} \begin{bmatrix} 4.84615 \\ 4.76923 \end{bmatrix} \cong \begin{bmatrix} 1.0000 \\ 0.98413 \end{bmatrix}$$

$$Ax_3 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1.00000 \\ 0.98413 \end{bmatrix} \simeq \begin{bmatrix} 4.96825 \\ 4.95238 \end{bmatrix}$$

$$x_4 = \frac{Ax_3}{\max(Ax_3)} = \frac{1}{4.96825} \begin{bmatrix} 4.96825 \\ 4.95238 \end{bmatrix} \simeq \begin{bmatrix} 1.00000 \\ 0.9968 \end{bmatrix}$$

$$Ax_4 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1.0000 \\ 0.9968 \end{bmatrix} \simeq \begin{bmatrix} \quad \end{bmatrix}$$

$$x_5 = \frac{Ax_4}{\max(Ax_4)} =$$

Singular Value Decomposition:

If A is $m \times n$ matrix and if $\lambda_1, \lambda_2, \ldots$ are the eigen value of $A^T A$.
Then the number $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$, ... are called the singular value of A.

Q: Find the singular value decomposition of the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Soln. First we find $A^T A$ is

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Now find the Eigen value

The characteristic of $A^T A$ is

$$|\lambda I - A^T A| = \begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{vmatrix} = 0$$

$\lambda = 3, 1$

so $\lambda_1 = 3$ & $\lambda_2 = 1$

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$, $\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$

The Eigen vector corresponding to these E.V

$$v_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} , \quad v_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Now, to find u_i where $u_i = \frac{1}{\sigma_i} A v_i$

so $u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

$$= \begin{bmatrix} \sqrt{6}/3 \\ \sqrt{6}/6 \\ \sqrt{6}/6 \end{bmatrix}$$

$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -1/\sqrt{3} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Now, The singular value of decomposition of A is

$$A = U \Sigma V^T$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/3 & 0 & -1/\sqrt{3} \\ \frac{\sqrt{6}}{6} & \frac{-\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$