SEC (8.1) GENERAL LINEAR TRANSFORMATIONS

Upto now our study of linear transformations has focused on transformations from R^n to R^m. In this Section, we will turn our attention to linear transformations involving general vector spaces. We will illustrate ways in which such transformations arise, and we will establish a fundamental relationship between general n-dimensional vector spaces and R^n.

Definitions and Terminology :

In Section 4.9 we defined a Matrix Transformation $T_A : R^n \to R^m$ to be a mapping of the form

$$T_A(\mathbf{x}) = A\mathbf{x}$$

in which A is an $m \times n$ matrix. We subsequently established that the matrix transformations are precisely the Linear Transformations from R^n to R^m, that is, the transformations with the linearity properties

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

$$\text{and} \quad T(k\mathbf{u}) = kT(\mathbf{u})$$

We will use these two properties for defining more general linear transformations —

DEFINITION

If $T: V \to W$ is a function from a vector space V to a vector space W, then T is called a Linear Transformation from V to W if the following two properties hold for all vectors $\mathbf{u}$ & $\mathbf{v}$ in V and for all scalars k —

(i) $T(k\mathbf{u}) = kT(\mathbf{u})$ [Homogeneity Property]

(ii) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ [Additivity Property]

In the special case where $V = W$, the linear transformation T is called a Linear Operator on the vector space V.

NOTE : The Homogeneity and Additivity properties of a linear transformation $T: V \to W$ can be used in combination to show that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors in V and k_1, k_2 are scalars, then

$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2)$$

More generally, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are vectors in V and $k_1, k_2, \ldots, k_r$ are any scalars, then

$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + \cdots + k_rT(\mathbf{v}_r) \quad \text{— (1)}$$

THEOREM (1) If $T: V \to W$ is a linear transformation, then

(i) $T(\mathbf{0}) = \mathbf{0}$.

(ii) $T(\mathbf{u}-\mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all $\mathbf{u}$ & $\mathbf{v}$ in V.

Example ① Matrix Transformations

Because we have based the defi. of a general linear transformation on the Homogeneity & Additivity properties of Matrix Transformations, it follows that a matrix transformation $T_A : R^n \to R^m$ is also a Linear Transformation in this more general sense with $V = R^n$ & $W = R^m$.

Example ② The Zero Transformation

Let V & W are any two vector spaces. The mapping $T: V \to W$ such that $T(\mathbf{v}) = \mathbf{0}, \forall \mathbf{v} \in V$ is a linear transformation, called the Zero Transformation.

To see that T is linear, observe that

$$T(u+v) = 0, \; T(u) = 0, \; T(v) = 0 \quad \& \; T(ku) = 0, \text{ by defi.}$$

Therefore,

$$T(u+v) = T(u) + T(v)$$

and

$$T(ku) = kT(u)$$

Example ③ The Identity Operator

Let V is any vector space. The mapping $I: V \to V$ defined by $I(\mathbf{v}) = \mathbf{v}$ is called the Identity Operator on V. We can verify that I is linear.

Example ④ Dilation and Contraction Operators

If V is a vector space and k is any scalar, then the mapping $T: V \to V$ given by $T(\mathbf{x}) = k\mathbf{x}$ is a linear operator on V.

Because if c is any scalar and if $\mathbf{u}$ & $\mathbf{v}$ are any vectors in V, then

$$\begin{aligned} T(c\mathbf{u}) &= k(c\mathbf{u}) \\ &= c(k\mathbf{u}) \\ \Rightarrow \quad T(c\mathbf{u}) &= cT(\mathbf{u}) \end{aligned}$$

and

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= k(\mathbf{u}+\mathbf{v}) \\ &= k\mathbf{u} + k\mathbf{v} \\ T(\mathbf{u}+\mathbf{v}) &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

If $0 < k < 1$, then T defined above is called the Contraction of V with factor k, and if $k > 1$, it is called the Dilation of V with factor k.

Example ⑤ A Linear Transformation from P_n to P_{n+1}

Let $p = p(x) = c_0 + c_1x + \cdots + c_nx^n$ be a polynomial in P_n, then the transformation $T: P_n \to P_{n+1}$ defined by $T(p) = T(p(x)) = xp(x) = c_0x + c_1x^2 + \cdots + c_nx^{n+1}$ is a linear transformation.

Because for any scalar k and any polynomials p_1 & p_2 in P_n, we have

$$\begin{aligned} T(kp) &= T(k\,p(x)) \\ &= x[kp(x)] \\ &= k[xp(x)] \\ T(kp) &= kT(p) \end{aligned}$$

and

$$\begin{aligned} T(p_1+p_2) &= T(p_1(x)+p_2(x)) \\ &= x[p_1(x)+p_2(x)] \\ &= xp_1(x)+xp_2(x) \\ T(p_1+p_2) &= T(p_1)+T(p_2) \end{aligned}$$

Example ⑥ A Linear Transformation Using an Inner Product

Let V be an inner product space & $\mathbf{v}_0$ be any fixed vector in V, then the transformation $T: V \to R$ defined by $T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v}_0 \rangle$ (that maps a vector $\mathbf{x}$ into its inner product with $\mathbf{v}_0$) is a linear transformation.

Because if k is any scalar, and if $\mathbf{u}$ & $\mathbf{v}$ are any vectors in V, then it follows from the properties of inner products that

$$\begin{aligned} T(k\mathbf{u}) &= \langle k\mathbf{u}, \mathbf{v}_0 \rangle \\ &= k\langle \mathbf{u}, \mathbf{v}_0 \rangle \\ T(k\mathbf{u}) &= kT(\mathbf{u}) \end{aligned}$$

and

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= \langle \mathbf{u}+\mathbf{v}, \mathbf{v}_0 \rangle \\ &= \langle \mathbf{u}, \mathbf{v}_0 \rangle + \langle \mathbf{v}, \mathbf{v}_0 \rangle \\ T(\mathbf{u}+\mathbf{v}) &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

Example ⑦ **Transformations on Matrix Spaces** :- Let M_{nn} be the vector space of all $n \times n$ matrices. In each part, determine whether the transformation is linear →

(i) $T_1(A) = A^T$ (ii) $T_2(A) = \det(A)$.

Soln. (i) It follows that

$$T_1(kA) = (kA)^T = kA^T$$

i.e; $T_1(kA) = kT_1(A)$

and $T_1(A+B) = (A+B)^T = A^T + B^T$

i.e; $T_1(A+B) = T_1(A) + T_1(B)$

Thus, T_1 is Linear.

(ii) It follows that $T_2(kA) = \det(kA)$

$$= k^n \det(A) = k^n T_2(A)$$

Thus, T_2 is not homogeneous and hence not Linear if $n > 1$.

Note that Additivity also fails because $\det(A+B)$ and $\det(A) + \det(B)$ are not generally equal.

Example ⑧ **Translation is Not Linear**

Part (i) of Theorem ① states that a linear transformation maps $\mathbf{0}$ to $\mathbf{0}$. This property is useful for identifying transformations that are not Linear.

For example, if $\mathbf{x}_0$ is a fixed non-zero vector in R^2, then the transformation

$$T(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$$

has the geometric effect of translating each point $\mathbf{x}$ in a direction parallel to $\mathbf{x}_0$ through a distance of $\|\mathbf{x}_0\|$.

This transformation cannot be Linear since $T(\mathbf{0}) = \mathbf{x}_0$, so T does not map $\mathbf{0}$ to $\mathbf{0}$.

FINDING LINEAR TRANSFORMATIONS FROM IMAGES OF BASIS VECTORS.

We saw in Sec 4.9 that if $T: R^n \to R^m$ is a matrix transformation, say multiplication by A, and if $e_1, e_2, \cdots, e_n$ are standard basis vectors for R^n, then A can be expressed as

$$A = [T(e_1) \mid T(e_2) \mid \cdots\cdots \mid T(e_n)]$$

It follows from this that the image of any vector $v = (c_1, c_2, \cdots, c_n)$ in R^n under multiplication by A can be expressed as

$$T(v) = c_1 T(e_1) + c_2 T(e_2) + \cdots + c_n T(e_n)$$

This formula tells us that for a matrix transformation, the image of any vector is expressible as a linear combination of the images of the standard basis vectors. This is a special case of the following more general result —

THEOREM ② Let $T: V \to W$ be a linear transformation, where V is finite dimensional. If $S = \{v_1, v_2, \cdots, v_n\}$ is a basis for V, then the image of any vector v in V can be expressed as

$$T(v) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$$

where $c_1, c_2, \cdots, c_n$ are the coefficients required to express v as a linear combination of vectors in S.

Example ⑨ Computing With Images of Basis Vectors

Consider the basis $S = \{v_1, v_2, v_3\}$ for R^3, where $v_1 = (1,1,1)$, $v_2 = (1,1,0)$, $v_3 = (1,0,0)$. Let $T: R^3 \to R^2$ be the linear transformation for which

$$T(v_1) = (1,0), \quad T(v_2) = (2,-1) \quad \& \quad T(v_3) = (4,3)$$

Find a formula for $T(x_1, x_2, x_3)$ and then use that formula to compute $T(2,-3,5)$.

Soln. We first need to express $x = (x_1, x_2, x_3)$ as a linear combination of v_1, v_2 & v_3.

Let $\quad x = c_1 v_1 + c_2 v_2 + c_3 v_3 \quad$ —— ①

$$(x_1, x_2, x_3) = c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$$

$$(x_1, x_2, x_3) = (c_1 + c_2 + c_3,\ c_1 + c_2,\ c_1)$$

Equating corresp. components on both sides,

$$c_1 + c_2 + c_3 = x_1 \quad \text{—— (i)}$$
$$c_1 + c_2 = x_2 \quad \text{—— (ii)}$$
$$c_1 = x_3 \quad \text{—— (iii)}$$

Solving (i), (ii) & (iii), we get $\quad c_1 = x_3, \quad c_2 = x_2 - x_3 \quad \& \quad c_3 = x_1 - x_2$

Hence from ①,
$$x = x_3 v_1 + (x_2 - x_3) v_2 + (x_1 - x_2) v_3$$

i.e.:
$$(x_1, x_2, x_3) = x_3 v_1 + (x_2 - x_3) v_2 + (x_1 - x_2) v_3$$

Thus
$$\begin{aligned}
T(x_1, x_2, x_3) &= x_3 T(v_1) + (x_2 - x_3) T(v_2) + (x_1 - x_2) T(v_3) \\
&= x_3(1,0) + (x_2 - x_3)(2,-1) + (x_1 - x_2)(4,3) \\
&= \big(x_3 + 2(x_2 - x_3) + 4(x_1 - x_2),\ 0 - (x_2 - x_3) + 3(x_1 - x_2)\big)
\end{aligned}$$

$$T(x_1, x_2, x_3) = (4x_1 - 2x_2 - x_3,\ 3x_1 - 4x_2 + x_3) \quad \text{—— ②}$$

From ②, we get $T(2,-3,5) = (4(2) - 2(-3) - 5,\ 3(2) - 4(-3) + 5) = (9, 23)$.

KERNEL AND RANGE : Recall that if A is an $m \times n$ matrix, then the Null space of A consists of all vectors $\mathbf{x}$ in R^n such that $A\mathbf{x} = 0$ and the column space of A consists of all vectors $\mathbf{b}$ in R^m for which there is at least one vector $\mathbf{x}$ in R^n such that $A\mathbf{x} = \mathbf{b}$. From the viewpoint of matrix transformations, the null space of A consists of all vectors in R^n that multiplication by A maps into $\mathbf{0}$, and the column space of A consists of all vectors in R^m that are images of at least one vector in R^n under multiplication by A. The fallowing definition extends these ideas to general linear transformations —

DEFINITION : If $T: V \to W$ is a linear transformation, then the set of vectors in V that T maps into $\mathbf{0}$ is called the Kernel of T and is denoted by $\ker(t)$. The set of all vectors in W that are images under T of at least one vector in V is called the Range of T and is denoted by $R(t)$.

Example (10) Kernel and Range of a Matrix Transformation

If $T_A: R^n \to R^m$ is multiplication by the $m \times n$ matrix A, then, the Kernel of T_A is the null space of A and the range of T_A is the column space of A.

Example (11) Kernel and Range of Zero Transformation

Let $T: V \to W$ be Zero Transformation. Since T maps every vector in V into $\mathbf{0}$, it follows that $\ker(t) = V$. Moreover, since $\mathbf{0}$ is the only image under T of vectors in V, it fallows that $R(t) = \{\mathbf{0}\}$.

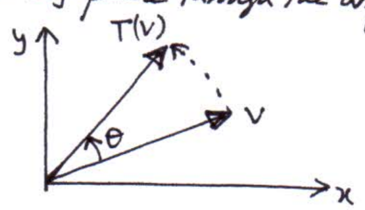
Example (12) Kernel and Range of Identity Operator

Let $I: V \to V$ be the Identity Operator. Since $I(\mathbf{v}) = \mathbf{v}$ for all vectors $\mathbf{v}$ in V, every vector in V is the image of some vector (namely, itself), thus $R(I) = V$.

Since the only vector that I maps into $\mathbf{0}$ is $\mathbf{0}$, it follows that $\ker(I) = \{\mathbf{0}\}$.

Example (13) Kernel and Range of Rotation

Let $T: R^2 \to R^2$ be the linear operator that rotates each vector in the xy-plane through the angle θ. (See Fig.). Since each vector in the xy-plane can be obtained by rotating some vector through the angle θ, it fallows that $R(t) = R^2$. Moreover, the only vector that rotates into $\mathbf{0}$ is $\mathbf{0}$, so $\ker(t) = \{\mathbf{0}\}$.



PROPERTIES OF KERNEL AND RANGE

THEOREM : If $T: V \to W$ is a linear transformation, then –

(i) The kernel of T is a subspace of V.

(ii) The range of T is a subspace of W.

RANK AND NULLITY OF LINEAR TRANSFORMATIONS

Definition : Let $T: V \to W$ be a linear transformation. If the range of T is finite-dimensional, then its dimension is called the Rank of T; and if the kernel of T is finite-dimensional, then its dimension is called the Nullity of T.

The rank of T is denoted by $rank(t)$ and the nullity of T by $nullity(t)$.

THEOREM. Dimension Theorem for Linear Transformations

If $T: V \to W$ is a linear transformation from an n-dimensional vector space V to a vector space W, then

$$rank(t) + nullity(t) = n$$

NOTE in the special case where A is $m \times n$ matrix and $T_A: R^n \to R^m$ is multiplication by A, the kernel of T_A is the null space of A, and the range of T_A is the column space of A. Thus, it follows from above Theorem that

$$rank(T_A) + nullity(T_A) = n.$$

SEC (8.2) ISOMORPHISM

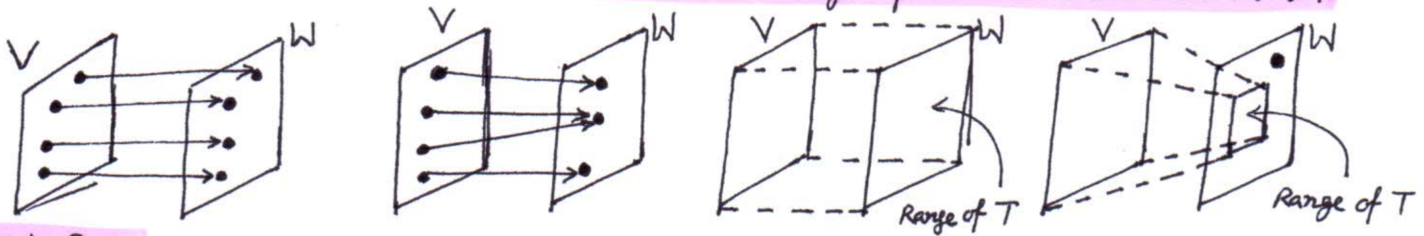
In this Section, we will establish a fundamental connection between real finite-dimensional vector spaces and the Euclidean space R^n. This connection is not only important theoretically, but it has practical applications in that it allows us to perform vector computations in general vector spaces by working with the vectors in R^n.

ONE-TO-ONE AND ONTO

Although many of the theorems in this text have been concerned exclusively with the vector space R^n, this is not as limiting as it might seem. As we will show, the vector space R^n is the 'mother' of all real n-dimensional vector spaces in the sense that any such space might differ from R^n in the notation used to represent vectors, but not in its algebraic structure. To explain what we mean by this, we will need two definitions —

Definition (1) If $T: V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be One-to-One if T maps distinct vectors in V into distinct vectors in W.

Definition (2) If $T: V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be Onto if every vector in W is the image of at least one vector in V.



One-to-One. Not One-to-One Onto W Not Onto W

The following theorem provides a useful way of telling whether a linear transformation is one-to-one by examining its kernel —

THEOREM (1) If $T: V \to W$ is a linear transformation, then the following statements are equivalent —

(i) T is One-to-One.

(ii) $\ker(t) = \{0\}$.

In the special case where V is finite-dimensional and T is a linear operator on V, then we can add a third statement to those in Theorem (1).

THEOREM (2) If V is a finite-dimensional vector space and if $T: V \to V$ is a linear operator, then the following statements are equivalent —

(i) T is One-to-One.

(ii) $\ker(t) = \{0\}$.

(iii) T is Onto [ie, $R(t) = V$].

Example ① Dilations and Contractions are One-to-One and Onto

Show that if V is a finite-dimensional vector space and c is any non-zero scalar, then the linear operator $T: V \to V$ defined by $T(\mathbf{v}) = c\mathbf{v}$ is one-to-one and onto.

Soln: If $\mathbf{v}$ is any vector in V (co-domain) then that vector is the image of vector $(\frac{1}{c})\mathbf{v}$ in V so the operator T is Onto.

Hence by Theorem ②, T is one-to-one.

Example ② Matrix Operators

If $T_A: R^n \to R^n$ is the matrix operator $T_A(x) = Ax$, then it follows (from Sec (5.1)) that T_A is one-to-one and onto if and only if matrix A is Invertible.

Example ③ Basic Transformations That are One-to-One and Onto

The linear transformations $T_1: P_3 \to R^4$ and $T_2: M_{22} \to R^4$ defined by

$$T_1(a + bx + cx^2 + dx^3) = (a, b, c, d)$$

$$\text{and} \quad T_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b, c, d)$$

are both one-to-one and onto. (We can verify by showing that their kernels contain only zero vector.)

Example ④ A One-to-One Linear Transformation

Let $T: P_n \to P_{n+1}$ be the linear transformation $T(\mathbf{p}) = T(p(x)) = xp(x)$ [discussed in example ⑤ of Sec (8.1)]

If $\mathbf{p} = p(x) = c_0 + c_1x + \cdots + c_nx^n$ and $\mathbf{q} = q(x) = d_0 + d_1x + \cdots + d_nx^n$ are distinct polynomials, then they differ in at least one coefficient.

Thus, $T(\mathbf{p}) = c_0x + c_1x^2 + \cdots + c_nx^{n+1}$ and $T(\mathbf{q}) = d_0x + d_1x^2 + \cdots + d_nx^{n+1}$ also differ in at least one coefficient.

It follows that T is one-to-one since it maps distinct polynomials $\mathbf{p}$ & $\mathbf{q}$ into distinct poly. $T(\mathbf{p})$ & $T(\mathbf{q})$

Example ⑤ A Transformation That is not One-to-One

Let $D: C^1(-\infty, \infty) \to F(-\infty, \infty)$ be the Differentiation transformation.

This linear transformation is not one-to-one because it maps functions that differ by a constant into the same function.

For example, $D(x^2) = D(x^2 + 1) = 2x$.

Dimension and Linear Transformations

There are two important facts about a linear transformation $T: V \to W$ in the case where V and W are finite-dimensional —

(i) If $\dim(W) < \dim(V)$, then T cannot be one-to-one.

(ii) If $\dim(V) < \dim(W)$, then T cannot be Onto.

stated informally, if a linear transformation maps a 'bigger' space to a 'smaller' space, then some points in the 'bigger' space must have the same image; and if a linear transformation maps a 'smaller' space to a 'bigger' space, then there must be points in the 'bigger' space that are not images of any points in the 'smaller' space.

Remark – These observations tells us, for example, that any linear transformation from R^3 to R^2 must map some distinct points of R^3 into the same point in R^2, and it also tells us that there is no linear transformation that maps R^2 onto all of R^3.

ISOMORPHISM

Defi. If a linear transformation $T: V \to W$ is both one-to-one and onto, then T is said to be an Isomorphism and the vector spaces V and W are said to be Isomorphic.

NOTE – The word 'Isomorphic' is derived from the Greek words iso, meaning "identical" and morphe, meaning "form". This terminology is appropriate because, as we will now explain, isomorphic vector spaces have the same "algebraic form", even though they may consists of different kinds of objects. To illustrate this idea, examine Table ① in which we have shown how the isomorphism

$$a_0 + a_1 x + a_2 x^2 \xrightarrow{T} (a_0, a_1, a_2)$$

matches up vector operations in P_2 and R^3.

Table ①

Operation in P_2	Operation in R^3
$3(1-2x+3x^2) = 3-6x+9x^2$	$3(1,-2,3) = (3,-6,9)$
$(2+x-x^2)+(1-x+5x^2) = 3+4x^2$	$(2,1,-1)+(1,-1,5) = (3,0,4)$
$(4+2x+3x^2)-(2-4x+3x^2) = 2+6x$	$(4,2,3)-(2,-4,3) = (2,6,0)$

The following theorem, which is one of the most important results in linear algebra, reveals the fundamental importance of the vector space R^n —

THEOREM ③ Every real n-dimensional vector space is Isomorphic to R^n.

NOTE – This theorem tells us that a real n-dimensional vecter space may differ from R^n in notation but its algebraic structure will be the same.

Example (6) The Natural Isomorphism from P_{n-1} to R^n

The mapping $\quad a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} \xrightarrow{T} (a_0, a_1, \cdots, a_{n-1})$

from P_{n-1} to R^n is one-to-one, onto and linear.

This is called Natural Isomorphism from P_{n-1} to R^n because, as the following computations show, it maps the natural basis $\{1, x, x^2, \cdots, x^{n-1}\}$ for P_{n-1} into standard basis for R^n:

$$1 = 1 + 0x + 0x^2 + \cdots + 0x^{n-1} \xrightarrow{T} (1, 0, 0, \cdots, 0)$$

$$x = 0 + 1x + 0x^2 + \cdots + 0x^{n-1} \xrightarrow{T} (0, 1, 0, \cdots, 0)$$

$$\vdots$$

$$x^{n-1} = 0 + 0x + 0x^2 + \cdots + 1.x^{n-1} \longrightarrow (0, 0, 0, \cdots, 1)$$

Example (7) The Natural Isomorphism from M_{22} to R^4

The matrices $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

form a basis for the vector space M_{22} of 2x2 matrices.

An isomorphism $T: M_{22} \to R^4$ can be constructed by first writing a matrix A in M_{22} in terms of the basis vectors as

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and then defining T as

$$T(A) = (a_1, a_2, a_3, a_4)$$

Thus, for example, $\begin{bmatrix} 1 & -3 \\ 4 & 6 \end{bmatrix} \xrightarrow{T} (1, -3, 4, 6)$.

NOTE – More generally, this idea can be used to show that the vector space M_{mn} of mxn matrices with real entries is isomorphic to R^{mn}.

SEC (8.3) COMPOSITIONS AND INVERSE TRANSFORMATIONS.

In Sec 4.10, we discussed compositions and inverses of matrix transformations. In this Section, we will extend some of those ideas to general linear transformations.

Composition of Linear Transformations

Definition: If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then the Composition of T_2 with T_1, denoted by $T_2 \circ T_1$, is the function defined by the formula

$$(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u})) \quad \text{——— } ①$$

where $\mathbf{u}$ is a vector in U.

NOTE: Observe that this definition requires that the domain of T_2 (which is V) contain the range of T_1. This is essential for the formula $T_2(T_1(\mathbf{u}))$ to make sense (fig.).

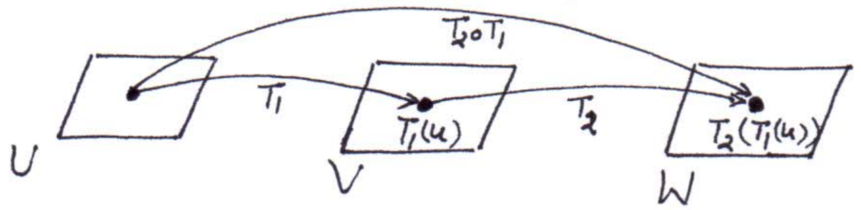


Fig. The Composition of T_2 with T_1

Our first theorem shows that the composition of two linear transformations is itself a linear transformation —

THEOREM ① If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then $(T_2 \circ T_1) : U \to W$ is also a linear transformation.

Example ① Composition of Linear Transformations

Let $T_1 : P_1 \to P_2$ and $T_2 : P_2 \to P_2$ are linear transformations given by

$$T_1(p(x)) = x p(x)$$

and $$T_2(p(x)) = p(2x+4)$$

Then the composition $(T_2 \circ T_1) : P_1 \to P_2$ is given by formula —

$$\begin{aligned}(T_2 \circ T_1)(p(x)) &= T_2(T_1(p(x))) \\ &= T_2(x p(x)) \\ &= (2x+4)\, p(2x+4)\end{aligned}$$

NOTE In particular, if $p(x) = c_0 + c_1 x$, then

$$\begin{aligned}(T_2 \circ T_1)(p(x)) = (T_2 \circ T_1)(c_0 + c_1 x) &= T_2[T_1(c_0 + c_1 x)] \\ &= T_2[x(c_0 + c_1 x)] \\ &= (2x+4)[c_0 + c_1(2x+4)]\end{aligned}$$

Example ② Composition with the Identity Operator

If $T: V \to V$ is any linear operator and if $I: V \to V$ is the Identity operator, then for all vectors **v** in V, we have

$$(T \circ I)(v) = T(I(v)) = T(v)$$

$$(I \circ T)(v) = I(T(v)) = T(v)$$

It follows that $T \circ I$ and $I \circ T$ are the same.

Compositions can be defined for more than two linear transformations as illustrated in Fig. For example, if $T_1: U \to V$, $T_2: V \to W$ and $T_3: W \to Y$ are linear transformations, then the composition $T_3 \circ T_2 \circ T_1$ is defined by —

$$(T_3 \circ T_2 \circ T_1)(u) = T_3(T_2(T_1(u))) \qquad \text{—— ②}$$

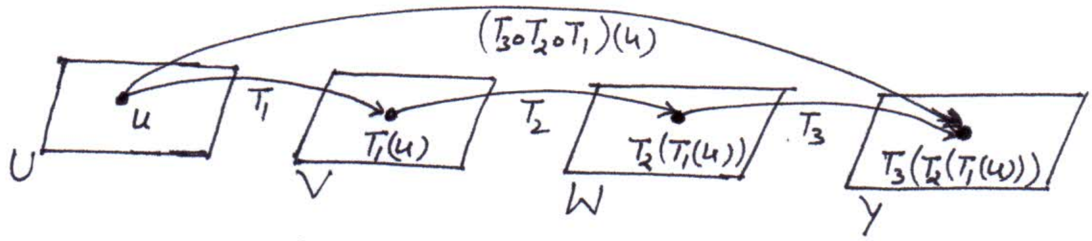


Fig. The Composition of three linear transformations.

INVERSE LINEAR TRANSFORMATIONS

Recall that if $T: V \to W$ is a linear transformation, then the range of T, denoted by $R(t)$, is the subspace of W consisting of all images under T of vectors in V.
If T is one-to-one, then each vector w in $R(t)$ is the image of a unique vector $\mathbf{v}$ in V. This uniqueness allows us to define a new function, called the Inverse of T and denoted by T^{-1}, that maps $\mathbf{w}$ back into $\mathbf{v}$ (fig.)

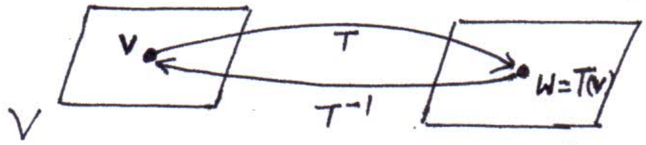


fig. The inverse of T maps $T(\mathbf{v})$ back into $\mathbf{v}$

It can be proved that $T^{-1}: R(t) \to V$ is a linear transformation. Moreover, it follows from the defi. of T^{-1} that

$$T^{-1}(T(\mathbf{v})) = T^{-1}(\mathbf{w}) = \mathbf{v} \quad —— (3)$$

$$T(T^{-1}(\mathbf{w})) = T(\mathbf{v}) = \mathbf{w} \quad —— (4)$$

so that T and T^{-1}, when applied in succession in either order, cancel the effect of each other.

NOTE: It is important to note that if $T: V \to W$ is a one-to-one linear transformation, then the domain of T^{-1} is the range of T, where the range may or may not be all of W. However, in the special case where $T: V \to V$ is a one-to-one Linear Operator and V is n-dimensional, then it follows that T must also be Onto, so the domain of T^{-1} is all of V.

Example (3) An Inverse Transformation

In Example (5) of Sec (8.2), we showed that the linear transformation $T: P_n \to P_{n+1}$ given by

$$T(p) = T(p(x)) = xp(x)$$

is one-to-one; thus T has an Inverse.
In this case the range of T is not all of P_{n+1} but rather the subspace of P_{n+1} consisting of polynomials with a zero constant term. This is evident from the formula for T:

$$T(c_0 + c_1x + \cdots + c_nx^n) = c_0x + c_1x^2 + \cdots + c_nx^{n+1}$$

It follows that $T^{-1}: R(t) \to P_n$ is given by the formula

$$T^{-1}(c_0x + c_1x^2 + \cdots + c_nx^{n+1}) = c_0 + c_1x + \cdots + c_nx^n$$

For example, in the case when $n \geq 3$,

$$T^{-1}(2x - x^2 + 5x^3 + 3x^4) = 2 - x + 5x^2 + 3x^3.$$

Example (4) An Inverse Transformation

Let $T: R^3 \to R^3$ be the linear operator defined by the formula

$$T(x_1, x_2, x_3) = (3x_1 + x_2, -2x_1 - 4x_2 + 3x_3, 5x_1 + 4x_2 - 2x_3)$$

Determine whether T is one-to-one; if so, find $T^{-1}(x_1, x_2, x_3)$.

Soln: We know that standard basis for R^3 is the set $B = \{e_1, e_2, e_3\}$, where

$$e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1)$$

Given that $T(x_1, x_2, x_3) = (3x_1 + x_2, -2x_1 - 4x_2 + 3x_3, 5x_1 + 4x_2 - 2x_3)$ ——(1)

$\therefore$ $T(e_1) = T(1,0,0) = (3,-2,5)$

$T(e_2) = T(0,1,0) = (1,-4,4)$

$T(e_3) = T(0,0,1) = (0,3,-2)$

Hence, standard matrix for T is $[T] = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

$$\det(T) = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix}$$

$$= 3(8-12) - (4-15) = -1$$

Hence the matrix T is invertible (and so operator T is one-to-one).

Now standard matrix for T^{-1} is

$$[T^{-1}] = [T]^{-1}$$

$$= \begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix} \qquad \left\{ \text{Find } [T]^{-1} \text{ using } \frac{1}{\det(T)} \operatorname{adj}(T) \right.$$

It follows that

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = [T^{-1}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 4x_1 - 2x_2 - 3x_3 \\ -11x_1 + 6x_2 + 9x_3 \\ -12x_1 + 7x_2 + 10x_3 \end{bmatrix}$$

Expressing this result in horizontal notation yields

$$T^{-1}(x_1, x_2, x_3) = (4x_1 - 2x_2 - 3x_3, -11x_1 + 6x_2 + 9x_3, -12x_1 + 7x_2 + 10x_3).$$

Composition of One-to-One Linear Transformations

The following theorem shows that a composition of one-to-one linear transformations is one-to-one and it relates inverse of a composition to inverses of its individual linear transformations.

THEOREM (2) If $T_1 : U \to V$ and $T_2 : V \to W$ are one-to-one linear transformations, then

(i) $T_2 \circ T_1$ is one-to-one.

(ii) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

NOTE: The result (ii) can be extended to compositions of three or more linear transformations for example, $(T_3 \circ T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1} \circ T_3^{-1}$

SEC (8.4) MATRICES FOR GENERAL LINEAR TRANSFORMATIONS.

In this Section, we will show that a general linear transformation from any n-dimensional vector space V to any m-dimensional vector space W can be performed using an appropriate matrix transformation from R^n to R^m. This idea is used in computer computations since computers are well suited for performing matrix computations.

Matrices of Linear Transformations

Suppose that V is an n-dimensional vector space, W is an m-dimensional vector space and that $T: V \to W$ is a linear transformation. Suppose further that B is a basis for V, that B' is a basis for W and that for each vector $\mathbf{x}$ in V, the co-ordinate matrices for x and $T(x)$ are $[x]_B$ and $[T(x)]_{B'}$, respectively (fig (i)).

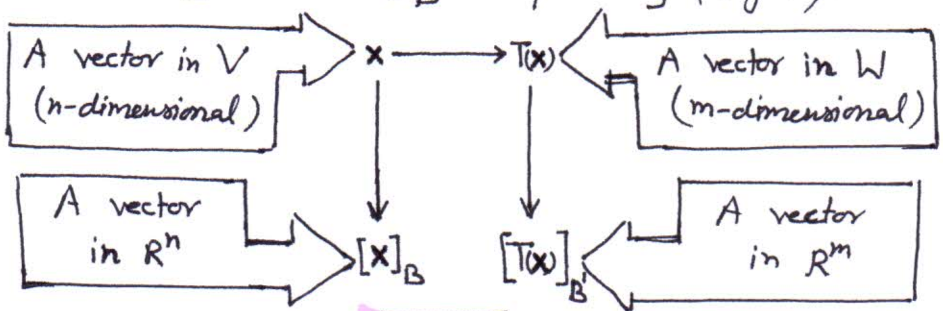


Fig (i)

It will be our goal to find an $m \times n$ matrix A such that multiplication by A maps the vector $[\mathbf{x}]_B$ into the vector $[T(\mathbf{x})]_{B'}$ for each x in V (Fig (ii) a). If we can do so, then as illustrated in fig (ii) b, we will be able to execute the linear transformation T by using matrix multiplication and the following indirect procedure –

Finding $T(x)$ Indirectly

Step ① Compute the co-ordinate vector $[x]_B$.

Step ② Multiply $[x]_B$ on the left by A to produce $[T(\mathbf{x})]_{B'}$.

Step ③ Reconstruct $T(\mathbf{x})$ from its co-ordinate vector $[T(\mathbf{x})]_{B'}$.

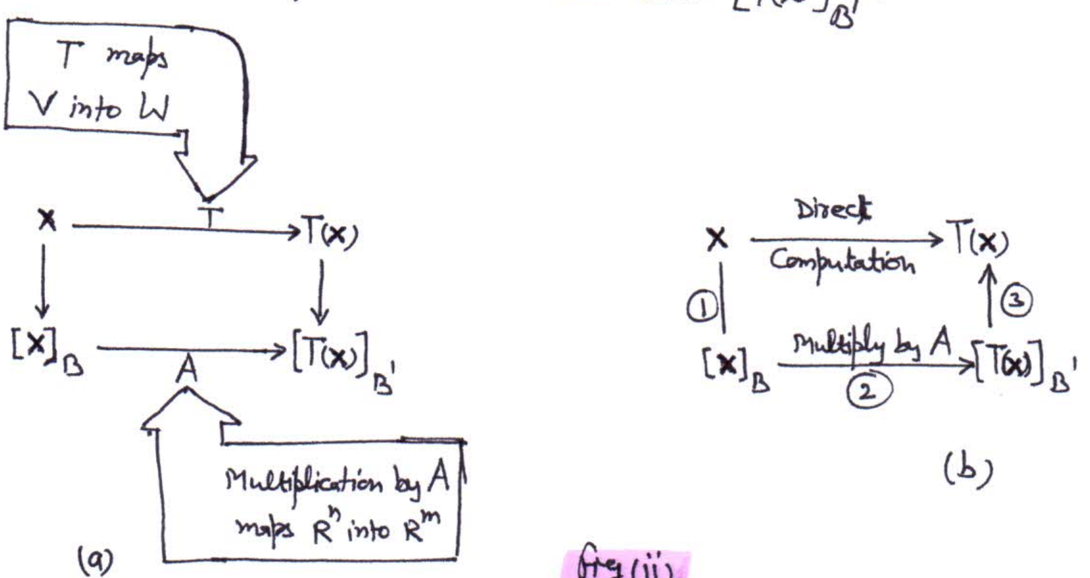


Fig (ii)

The key to execute this plan is to find an $m \times n$ matrix A with the property that

$$A[\mathbf{x}]_B = [T(\mathbf{x})]_{B'} \quad \text{——— (1)}$$

For this purpose, let $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be a basis for n-dimensional space V and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is a basis for m-dimensional space W.

The matrix for T relative to the bases B and B', denoted by $[T]_{B',B}$ is

$$[T]_{B',B} = \Big[[T(\mathbf{u}_1)]_{B'} \quad [T(\mathbf{u}_2)]_{B'} \quad \cdots \quad [T(\mathbf{u}_n)]_{B'} \Big] \quad \text{—— (2)}$$

and from (1), this matrix has the property

$$[T]_{B',B} [\mathbf{x}]_B = [T(\mathbf{x})]_{B'} \quad \text{——— (3)}$$

In the special case where $T_A : R^n \to R^m$ is multiplication by A and where B & B' are the standard bases for R^n and R^m, respectively, then

$$[T]_{B',B} = A \quad \text{——— (4)}$$

Example (1) **Matrix for a Linear Transformation**

Let $T : P_1 \to P_2$ be the linear transformation defined by $T(p(x)) = xp(x)$

Find the matrix for T with respect to standard bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{u}_1 = 1$, $\mathbf{u}_2 = x$, $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x$, $\mathbf{v}_3 = x^2$.

Soln. From the given formula for T, we obtain

$$T(u_1) = T(1) = x(1) = x$$

$$T(u_2) = T(x) = x(x) = x^2$$

Writing $T(u_1)$ in terms of $v_1, v_2,$ & v_3 as —

$$T(u_1) = x = 0.1 + 1.x + 0.x^2$$

$$T(u_1) = 0v_1 + 1v_2 + 0v_3$$

$\therefore$ Coordinate vector for $T(u_1)$ relative to basis B' is $[T(u_1)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Writing $T(u_2)$ in terms of $v_1, v_2,$ & v_3 as —

$$T(u_2) = x^2 = 0.1 + 0.x + 1.x^2$$

$$T(u_2) = 0v_1 + 0v_2 + 1v_3$$

$\therefore$ Co-ordinate vector for $T(u_2)$ relative to basis B' is $[T(u_2)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Thus, matrix for T with respect to B & B' is

$$[T]_{B',B} = \Big[[T(u_1)]_{B'} \quad [T(u_2)]_{B'} \Big] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example ② Matrix for a Linear Transformation

Let $T: R^2 \to R^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1+13x_2 \\ -7x_1+16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the matrix for the transformation T with respect to the bases $B = \{u_1, u_2\}$ for R^2 and $B' = \{v_1, v_2, v_3\}$ for R^3, where

$$u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}; \qquad v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Soln: From the formula for T, we have

$$T(u_1) = T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$$

$$T(u_2) = T\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

Expressing these vectors as linear combinations of v_1, v_2, and v_3

Let $\quad T(u_1) = a_1 v_1 + a_2 v_2 + a_3 v_3 \qquad$ —— ①

$$(1,-2,-5) = a_1(1,0,-1) + a_2(-1,2,2) + a_3(0,1,2)$$

$$= (a_1, 0, -a_1) + (-a_2, 2a_2, 2a_2) + (0, a_3, 2a_3)$$

$$(1,-2,-5) = (a_1 - a_2,\ 2a_2 + a_3,\ -a_1 + 2a_2 + 2a_3)$$

Equating both sides, we get

$a_1 - a_2 = 1 \qquad$ —— (i)

$2a_2 + a_3 = -2 \qquad$ —— (ii)

$-a_1 + 2a_2 + 2a_3 = -5 \qquad$ —— (iii)

Solving (i), (ii) & (iii), we get $a_1 = 1$, $a_2 = 0$ & $a_3 = -2$

∴ from ①, $\quad T(u_1) = v_1 + 0v_2 - 2v_3 \qquad$ —— ②

Thus $\quad [T(u_1)]_{B'} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \qquad$ —— ③

Similarly, we can show that $\quad T(u_2) = 3v_1 + v_2 - v_3$

Thus $\quad [T(u_2)]_{B'} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \qquad$ —— ④

Hence $\quad [T]_{B',B} = \begin{bmatrix} [T(u_1)]_{B'} & [T(u_2)]_{B'} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Matrices of Linear Operators

In the special case where $V = W$ (so that $T: V \to V$ is a linear operator), it is usual to take $B = B'$ when constructing a matrix for T. In this case, the resulting matrix is called the matrix for T relative to the basis B and is usually denoted by $[T]_B$ rather than $[T]_{B,B}$.

If $B = \{u_1, u_2, \cdots, u_n\}$ then

$$[T]_B = \Big[[T(u_1)]_B \quad [T(u_2)]_B \quad \cdots \quad [T(u_n)]_B\Big] \qquad \text{—— ①}$$

and $$[T]_B [x]_B = [T(x)]_B \qquad \text{—— ②}$$

In the special case where $T: R^n \to R^n$ is a matrix operator, say multiplication by 'A', and B is the standard basis for R^n, then formula ① simplifies to

$$[T]_B = A \qquad \text{—— ③}$$

Matrices of Identity Operators: Recall that the identity operator $I: V \to V$ maps every vector in V into itself, that is, $I(x) = x$ for every vector x in V. The following example shows that if V is n-dimensional, then the matrix for I relative to any basis B for V is the $n \times n$ identity matrix.

Example ③ Matrices of Identity Operators

Let $B = \{u_1, u_2, \cdots, u_n\}$ is a basis for a finite-dimensional vector space V, and if $I: V \to V$ is the identity operator on V, then

$$I(u_1) = u_1 = 1u_1 + 0u_2 + \cdots + 0u_n$$
$$I(u_2) = u_2 = 0u_1 + 1u_2 + \cdots + 0u_n$$
$$\vdots$$
$$I(u_n) = u_n = 0u_1 + 0u_2 + \cdots + 1u_n$$

Therefore, $$[I]_B = \Big[[I(u_1)]_B \quad [I(u_2)]_B \quad \cdots [I(u_n)]_B\Big]$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Example (4) Linear Operator on P_2

Let $T: P_2 \to P_2$ be the linear operator defined by

$$T(p(x)) = p(3x-5)$$

ie:, $T(c_0+c_1x+c_2x^2) = c_0 + c_1(3x-5) + c_2(3x-5)^2$

(i) Find $[T]_B$ relative to the basis $B = \{1, x, x^2\}$.

(ii) Use the indirect procedure to compute $T(1+2x+3x^2)$.

(iii) Check the result in (ii) by computing $T(1+2x+3x^2)$ directly.

Soln. (i) From the formula for T,

$$T(1) = 1 = 1 + 0x + 0x^2$$

$$T(x) = 3x-5 = -5+3x+0x^2$$

$$T(x^2) = (3x-5)^2 = 9x^2 - 30x + 25 = 25 - 30x + 9x^2$$

Thus, $[T]_B = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}$

(ii) Step (1) The coordinate matrix for $p = 1+2x+3x^2$ relative to basis $B = \{1, x, x^2\}$ is

$$[p]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Step (2) Multiplying $[p]_B$ by the matrix $[T]_B$ found in part (i), we obtain

$$[T]_B[p]_B = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 66 \\ -84 \\ 27 \end{bmatrix} = [T(p)]_B$$

Step (3) Reconstructing $T(p) = T(1+2x+3x^2)$ from $[T(p)]_B$, we obtain

$$T(1+2x+3x^2) = 66 - 84x + 27x^2$$

(iii) By direct computation,

$$T(1+2x+3x^2) = 1 + 2(3x-5) + 3(3x-5)^2$$

$$= 1 + 6x - 10 + 3(9x^2 + 25 - 30x)$$

$$= 66 - 84x + 27x^2$$

Which agrees with the result in (ii).