SEC (4.1) REAL VECTOR SPACES

In this Section, we will extend the concept of a vector by using the basic properties of vectors in R^n as axioms, which if satisfied by a set of objects, guarantee that those objects behave like familiar vectors.

VECTOR SPACE AXIOMS -- The following definition consists of ten axioms, eight of which are properties of vectors in R^n that were stated in theorem of Sec (3.1).

Definition : Let V be an arbitrary non-empty set of objects on which two operations are defined -- Addition, and Multiplication by scalars.

By Addition, we mean a rule for associating with each pair of objects $\mathbf{u}$ & $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the Sum of $\mathbf{u}$ & $\mathbf{v}$;

By Scalar Multiplication, we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called Scalar Multiple of $\mathbf{u}$ by k.

If the following axioms are satisfied by all objects $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars k and m, then we call V a vector space and we call objects in V as vectors.

(1) If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.

(2) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(3) $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$

(4) There is an object $\mathbf{0}$ in V, called zero vector for V, such that

$$\mathbf{0}+\mathbf{u} = \mathbf{u}+\mathbf{0} = \mathbf{u} \text{ for all } \mathbf{u} \text{ in } V$$

(5) For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called negative of $\mathbf{u}$, such that

$$\mathbf{u}+(-\mathbf{u}) = (-\mathbf{u})+\mathbf{u} = \mathbf{0}.$$

(6) If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.

(7) $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$

(8) $(k+m)\mathbf{u} = k\mathbf{u}+m\mathbf{u}$

(9) $k(m\mathbf{u}) = (km)\mathbf{u}$

(10) $1.\mathbf{u} = \mathbf{u}$

Our first example is the simplest of all vector spaces in that it contains only one object. Since Axiom ④ requires that every vector space contain a zero vector, the object will have to be that vector.

Example ① (The Zero Vector Space)

Let V consist of a single object, which we denote by $\mathbf{0}$ and define

$$\mathbf{0}+\mathbf{0}=\mathbf{0}$$

$$\text{and} \quad k\mathbf{0}=\mathbf{0} \quad \text{for all scalars } k.$$

It is easy to check that all vector axioms are satisfied. We call this the Zero Vector Space.

Example ② (R^n is a Vector Space)

Let $V=R^n$, and define the vector space operations on V to be the usual operations of addition and scalar multiplication of n-tuples; that is,

$$\mathbf{u}+\mathbf{v}=(u_1,u_2,\dots,u_n)+(v_1,v_2,\dots,v_n)$$
$$=(u_1+v_1,u_2+v_2,\dots,u_n+v_n)$$
$$\& \quad k\hat{\mathbf{u}}=(ku_1,ku_2,\dots,ku_n)$$

The set $V=R^n$ is closed under Addition and Scalar multiplication because the foregoing operations produce n-tuples as their end result.

Axiom ②
$$\mathbf{u}+\mathbf{v}=(u_1,u_2,\dots,u_n)+(v_1,v_2,\dots,v_n)$$
$$=(u_1+v_1,u_2+v_2,\dots,u_n+v_n)$$
$$=(v_1+u_1,v_2+u_2,\dots,v_n+u_n)$$
$$=(v_1,v_2,\dots,v_n)+(u_1,u_2,\dots,u_n)$$
$$=\mathbf{v}+\mathbf{u}$$

Axiom ③
$$\mathbf{u}+(\mathbf{v}+\mathbf{w})=(u_1,u_2,\dots,u_n)+\left[(v_1,v_2,\dots,v_n)+(w_1,w_2,\dots,w_n)\right]$$
$$=(u_1,u_2,\dots,u_n)+(v_1+w_1,v_2+w_2,\dots,v_n+w_n)$$
$$=\left[u_1+(v_1+w_1),u_2+(v_2+w_2),\dots,u_n+(v_n+w_n)\right]$$
$$=\left[(u_1+v_1)+w_1,(u_2+v_2)+w_2,\dots,(u_n+v_n)+w_n\right]$$
$$=(u_1+v_1,u_2+v_2,\dots,u_n+v_n)+(w_1,w_2,\dots,w_n)$$
$$=\left[(u_1,u_2,\dots,u_n)+(v_1,v_2,\dots,v_n)\right]+(w_1,w_2,\dots,w_n)$$
$$=(\mathbf{u}+\mathbf{v})+\mathbf{w}$$

Axiom ④ $\forall\ \mathbf{u}=(u_1,u_2,\dots,u_n)\in V$, we have
$$\mathbf{u}+\mathbf{0}=(u_1,u_2,\dots,u_n)+(0,0,\dots,0)$$
$$=(u_1+0,u_2+0,\dots,u_n+0)$$
$$=(u_1,u_2,\dots,u_n)$$
$$=\mathbf{u}$$

Axiom ⑤ $\forall\, \mathbf{u} = (u_1, u_2, \cdots, u_n) \in V$, we have $-\mathbf{u} = (-u_1, -u_2, \cdots, -u_n)$ such that

$$\begin{aligned}
\mathbf{u} + (-\mathbf{u}) &= (u_1, u_2, \cdots, u_n) + (-u_1, -u_2, \cdots, -u_n) \\
&= (u_1 - u_1, u_2 - u_2, \cdots, u_n - u_n) \\
&= (0, 0, \cdots, 0) = \mathbf{0}
\end{aligned}$$

Axiom ⑦

$$\begin{aligned}
k(\mathbf{u} + \mathbf{v}) &= k\left[(u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n)\right] \\
&= k(u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n) \\
&= \left[k(u_1 + v_1), k(u_2 + v_2), \cdots, k(u_n + v_n)\right] \\
&= (ku_1 + kv_1, ku_2 + kv_2, \cdots, ku_n + kv_n) \\
&= (ku_1, ku_2, \cdots, ku_n) + (kv_1, kv_2, \cdots, kv_n) \\
&= k(u_1, u_2, \cdots, u_n) + k(v_1, v_2, \cdots, v_n) \\
&= k\mathbf{u} + k\mathbf{v}
\end{aligned}$$

Axiom ⑧

$$\begin{aligned}
(k+m)\mathbf{u} &= (k+m)(u_1, u_2, \cdots, u_n) \\
&= \left((k+m)u_1, (k+m)u_2, \cdots, (k+m)u_n\right) \\
&= (ku_1 + mu_1, ku_2 + mu_2, \cdots, ku_n + mu_n) \\
&= (ku_1, ku_2, \cdots, ku_n) + (mu_1, mu_2, \cdots, mu_n) \\
&= k(u_1, u_2, \cdots, u_n) + m(u_1, u_2, \cdots, u_n) \\
&= k\mathbf{u} + m\mathbf{u}
\end{aligned}$$

Axiom ⑨

$$\begin{aligned}
k(m\mathbf{u}) &= k\left[m(u_1, u_2, \cdots, u_n)\right] \\
&= k(mu_1, mu_2, \cdots, mu_n) \\
&= (kmu_1, kmu_2, \cdots, kmu_n) \\
&= km(u_1, u_2, \cdots, u_n) \\
&= (km)\mathbf{u}
\end{aligned}$$

Axiom ⑩

$$\begin{aligned}
1.\mathbf{u} &= 1.(u_1, u_2, \cdots, u_n) \\
&= (1.u_1, 1.u_2, \cdots, 1.u_n) \\
&= (u_1, u_2, \cdots, u_n) \\
&= \mathbf{u}.
\end{aligned}$$

Example ③ (The Vector Space of Infinite Sequences of Real Numbers) ie. R^{∞}

Let V consists of objects of the form $\mathbf{u} = (u_1, u_2, \cdots, u_n, \cdots)$ in which $u_1, u_2, \cdots, u_n, \cdots$ is an infinite sequence of real numbers.

We define Addition and Scalar multiplication componentwise by

$$\mathbf{u}+\mathbf{v} = (u_1, u_2, \cdots, u_n, \cdots) + (v_1, v_2, \cdots, v_n, \cdots)$$
$$= (u_1+v_1, u_2+v_2, \cdots, u_n+v_n, \cdots)$$

& $$k\mathbf{u} = (ku_1, ku_2, \cdots, ku_n, \cdots)$$

This is left as an exercise to confirm that V with these operations is a Vector Space. We will denote this vector space by the symbol R^{∞}.

Example ④ (A Vector Space of 2x2 Matrices)

Let V is the set of 2x2 matrices with real entries, and take the vector space operations on V to be the usual operations of matrix addition and scalar multiplication; that is,

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$

& $$k\mathbf{u} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

The set V is closed under Addition and Scalar multiplication because the foregoing operations produce 2x2 matrices as the end result.

Thus, it remains to confirm that Axioms 2, 3, 4, 5, 7, 8, 9 and 10 hold.

Axiom ②

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$
$$= \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$
$$= \begin{bmatrix} v_{11}+u_{11} & v_{12}+u_{12} \\ v_{21}+u_{21} & v_{22}+u_{22} \end{bmatrix}$$
$$= \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{v}+\mathbf{u}$$

Axiom ③

$$\mathbf{u}+(\mathbf{v}+\mathbf{w}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11}+w_{11} & v_{12}+w_{12} \\ v_{21}+w_{21} & v_{22}+w_{22} \end{bmatrix}$$
$$= \begin{bmatrix} u_{11}+(v_{11}+w_{11}) & u_{12}+(v_{12}+w_{12}) \\ u_{21}+(v_{21}+w_{21}) & u_{22}+(v_{22}+w_{22}) \end{bmatrix}$$
$$= \begin{bmatrix} (u_{11}+v_{11})+w_{11} & (u_{12}+v_{12})+w_{12} \\ (u_{21}+v_{21})+w_{21} & (u_{22}+v_{22})+w_{22} \end{bmatrix}$$
$$= \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$
$$= (\mathbf{u}+\mathbf{v})+\mathbf{w}$$

Axiom 4 for each $\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ in V, we have $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in V

such that $\mathbf{0} + \mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$

and similarly $\mathbf{u} + \mathbf{0} = \mathbf{u}$

Axiom 5 for each ~~abject~~ object $\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ in V, we have $-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$ in V

such that
$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$$
$$= \begin{bmatrix} u_{11} - u_{11} & u_{12} - u_{12} \\ u_{21} - u_{21} & u_{22} - u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

and similarly $(-\mathbf{u}) + \mathbf{u} = \mathbf{0}$

Axiom 7
$$k(\mathbf{u}+\mathbf{v}) = k \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$
$$= \begin{bmatrix} ku_{11}+kv_{11} & ku_{12}+kv_{12} \\ ku_{21}+kv_{21} & ku_{22}+kv_{22} \end{bmatrix}$$
$$= \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} + \begin{bmatrix} kv_{11} & kv_{12} \\ kv_{21} & kv_{22} \end{bmatrix}$$
$$= k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + k \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$
$$= k\mathbf{u} + k\mathbf{v}$$

Axiom 8
$$(k+m)\mathbf{u} = (k+m) \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$
$$= \begin{bmatrix} (k+m)u_{11} & (k+m)u_{12} \\ (k+m)u_{21} & (k+m)u_{22} \end{bmatrix}$$
$$= \begin{bmatrix} ku_{11}+mu_{11} & ku_{12}+mu_{12} \\ ku_{21}+mu_{21} & ku_{22}+mu_{22} \end{bmatrix}$$
$$= \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} + \begin{bmatrix} mu_{11} & mu_{12} \\ mu_{21} & mu_{22} \end{bmatrix}$$
$$= k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + m \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = k\mathbf{u} + m\mathbf{u}$$

Axiom 9
$$k(m\mathbf{u}) = k \begin{bmatrix} mu_{11} & mu_{12} \\ mu_{21} & mu_{22} \end{bmatrix} = (km) \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = (km)\mathbf{u}$$

Axiom 10
$$1.\mathbf{u} = 1 \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} 1.u_{11} & 1.u_{12} \\ 1.u_{21} & 1.u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

Example ⑤ The Vector Space of $m \times n$ Matrices ie:, M_{mn}

Example ④ is a special case of a more general class of vector spaces. You should have no trouble adapting the argument used in that example to show that the set V of all $m \times n$ matrices with the usual matrix operations of Addition and Scalar multiplication is a Vector Space. We will denote this vector space by the symbol M_{mn}. Thus, for example, the vector space in Example ④ is denoted as M_{22}.

Example ⑥ The Vector Space of Real-Valued Functions ie:, $F(-\infty,\infty)$

Let V be the set of real-valued functions that are defined at each x in the interval $(-\infty,\infty)$. If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two functions in V and if k is any scalar, then define the operations of Addition and Scalar multiplication by

$$(\mathbf{f}+\mathbf{g})(x) = f(x) + g(x) \quad \text{——— ①}$$

$$(k\mathbf{f})(x) = k f(x) \quad \text{——— ②}$$

One way to think about these operations is to view the numbers $f(x)$ and $g(x)$ as 'components' of $\mathbf{f}$ and $\mathbf{g}$ at the point x, in which case Equations ① and ② state that two functions are added by adding corresponding components, and a function is multiplied by a scalar by multiplying each component by that scalar – exactly as in R^n & R^∞. The set V with these operations is denoted by the symbol $F(-\infty,\infty)$.

NOTE. It is important to recognize that you cannot impose any two operations on any set V and expect the vector space axioms hold. For example,

If V is the set of n-tuples with positive components, and if the standard operations from R^n are used, then V is not closed under Scalar multiplication, because if $\mathbf{u}$ is a non-zero n-tuple in V, then $(-1)\mathbf{u}$ has at least one negative component and hence is not in V. The following is a less obvious example in which only one of the ten vector space axioms fails to hold.

Example ⑦ A Set That is Not A VECTOR SPACE

Let $V = R^2$ and define Addition and Scalar multiplication operations as follows –

If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define $\quad \mathbf{u}+\mathbf{v} = (u_1+v_1, u_2+v_2)$

and if k is any real no., then define $\quad k\mathbf{u} = (ku_1, 0)$

The Addition operation is the standard one from R^2 but scalar multiplication is not. We can see that first nine vector space axioms are satisfied. However, Axiom ⑩ fails to hold for certain vectors. For example, if $\mathbf{u} = (u_1, u_2)$ is such that $u_2 \neq 0$, then

$$1\mathbf{u} = 1(u_1, u_2) = (1.u_1, 0) = (u_1, 0) \neq \mathbf{u}$$

Thus V is not a vector space with the stated operations.

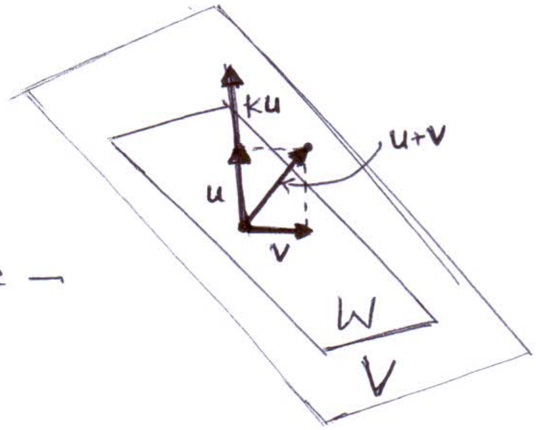
SEC (4.2) SUBSPACES

It is possible for one vector space to be contained within another. We will explore this idea in this Section, we will discuss how to recognize such vector spaces and we will give a variety of examples that will be used in our later work.

<u>Definition</u>. A subset W of a vector space V is called a Subspace of V if W is itself a vector space under the Addition and Scalar multiplication defined on V.

<u>NOTE</u>: In general, to show that a non-empty set W with two operations is a Vector Space, one must verify the ten vector space axioms. However, if W is a subspace of a known vector space V, then certain axioms need not be verified because they are 'inherited' from V. For example, it is not necessary to verify that $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ holds in W because it holds for all vectors in V including those in W.

On the other hand, it is necessary to verify that W is closed under Addition and Scalar multiplication since it is possible that adding two vectors in W or multiplying a vector in W by a scalar produces a vector in V that is outside of W (See fig.)



Those axioms that are not inherited by W are —

<u>Axiom ①</u> Closure of W under Addition

<u>Axiom ④</u> Existence of zero vector in W

<u>Axiom ⑤</u> Existence of a negative in W for every vector in W

<u>Axiom ⑥</u> Closure of W under Scalar multiplication

So these must be verified to prove that it is a subspace of V. However, the following theorem shows that if Axiom ① and Axiom ⑥ hold in W, then Axioms ④ and ⑤ hold in W as a consequence and hence need not to be verified.

<u>THEOREM</u>. If W is a set of one or more vectors in a vector space V, then W is a Subspace of V iff the following conditions hold —

(i) If $\mathbf{u}$ & $\mathbf{v}$ are vectors in W, then $\mathbf{u}+\mathbf{v}$ is in W.

(ii) If k is any scalar and $\mathbf{u}$ is any vector in W, then $k\mathbf{u}$ is in W.

<u>NOTE ①</u> In other words, the above theorem states that W is a Subspace of a vector space V iff it is closed under Addition and Scalar multiplication.

<u>NOTE ②</u> Every vector space has at least two subspaces, itself and its zero subspace

Example ① (The Zero Subspace)

If V is any vector space and if $W = \{\mathbf{0}\}$ is a subset of V that consists of the zero vector only then W is closed under Addition and Scalar multiplication since

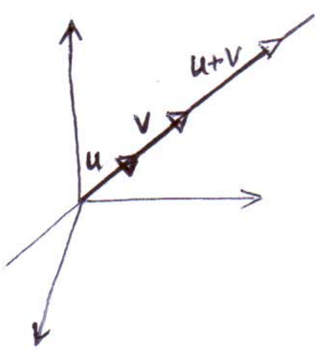
$$\mathbf{0} + \mathbf{0} = \mathbf{0}$$

$$\text{and} \quad k\mathbf{0} = \mathbf{0} \quad \text{for any scalar } k.$$

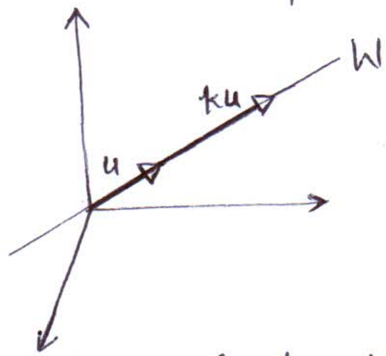
We call $W = \{\mathbf{0}\}$, the Zero Subspace of V.

Example ② (Lines Through the Origin Are Subspaces of R^2 and of R^3)

If W is a line through the origin of either R^2 or R^3, then adding two vectors on the line W or multiplying on the line W by a scalar produces another vector on the line W, so W is closed under Addition and Scalar multiplication



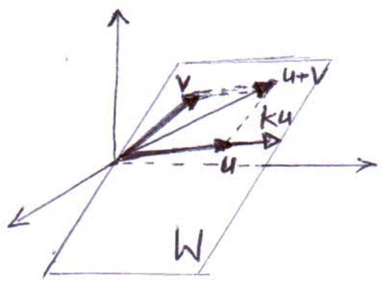
(i) W is closed under Addition



(ii) W is closed under Scalar multiplication

Example ③ (Planes Through the Origin Are Subspaces of R^3)

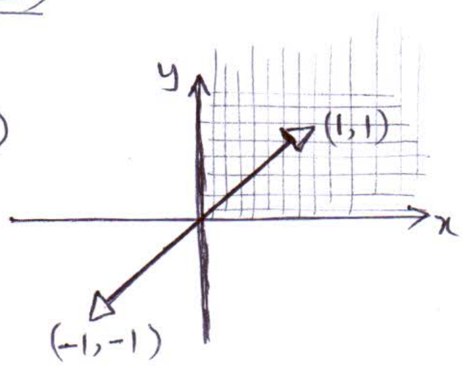
If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a plane W through the origin of R^3, then it is evident geometrically that $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ lie in the same plane W for any scalar k (See fig.). Thus W is closed under Addition and Scalar multiplication.



Example ④ (A Subset of R^2 That is Not a Subspace)

Let W be the set of all points (x, y) in R^2 for which $x \geq 0$ and $y \geq 0$ (the shaded region in fig.) This set is not a Subspace of R^2 because it is not closed under Scalar multiplication.

For example, $\mathbf{v} = (1,1)$ is a vector in W, but $(-1)\mathbf{v} = (-1,-1)$ is not a vector in W.



Example 5 (Subspaces of M_{nn})

We know from Sec (1.7) that the Sum of two symmetric $n\times n$ matrices is symmetric and that a scalar multiple of a symmetric $n\times n$ matrix is symmetric.

Thus, the set of symmetric $n\times n$ matrices is closed under Addition and Scalar multiplication and hence is a subspace of M_{nn}.

Similarly, the set of upper triangular matrices, lower triangular matrices and diagonal matrices are subspaces of M_{nn}.

Example 6 (A Subset of M_{nn} That is Not a Subspace)

The set W of invertible $n\times n$ matrices is not a Subspace of M_{nn}, failing on two counts — It is not closed under Addition and not closed under Scalar multiplication.

We will illustrate this with an example in M_{22} that can readily adapt to M_{nn}.

Consider the matrices $U = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $V = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$

Now $U+V = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 5 \end{bmatrix}$, which is not invertible

& $0U = 0\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which is not invertible.

Example 7 The Subspace $C(-\infty,\infty)$

There is a theorem in Calculus which states that a sum of continuous functions is continuous and that a constant times a continuous function is continuous.

Rephrased in vector language, the set of continuous functions on $(-\infty,\infty)$ is a Subspace of vector space of Real-Valued functions, $F(-\infty,\infty)$. We denote this subspace by $C(-\infty,\infty)$.

Example 8 (Functions with Continuous Derivatives)

A function with a continuous derivative is said to be continuously differentiable. There is a theorem is Calculus which states that the sum of two continuously differentiable functions is continuously differentiable and that a constant times a continuously differentiable function is continuously differentiable.

Thus, the functions that are continuously differentiable on $(-\infty,\infty)$ form a Subspace of $F(-\infty,\infty)$. We will denote this subspace by $C^1(-\infty,\infty)$, where the superscript emphasizes that the first derivative is Continuous.

To take this a further step, the set of functions with m continuous derivatives on $(-\infty,\infty)$ is a Subspace of $F(-\infty,\infty)$ as is the set of functions with derivatives of all orders on $(-\infty,\infty)$. We will denote these subspaces by $C^m(-\infty,\infty)$ and $C^\infty(-\infty,\infty)$ respectively.

Example (9) The Subspace of All Polynomials i.e. P_∞

Recall that a Polynomial is a function that can be expressed in the form

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \qquad \text{——— (1)}$$

where $a_0, a_1, \ldots, a_n$ are constants.

It is evident that the sum of two polynomials is a polynomial and that a constant times a polynomial is a polynomial.

Thus, the set W of all polynomials is closed under Addition and Scalar multiplication and hence is a Subspace of $F(-\infty,\infty)$. We will denote this subspace by P_∞.

Example (10) The Subspace of Polynomials of Degree $\le n$ i.e., P_n

Recall that the Degree of a Polynomial is the highest power of the variable that occurs with a non-zero coefficient. Thus, if $a_n \neq 0$ in eqn. (1), then that polynomial has degree n.

It is not True that the set W of polynomials with positive degree n is a Subspace of $F(-\infty,\infty)$ because the set is not closed under Addition. for example,

the polynomials $1+2x+3x^2$ and $5+7x-3x^2$ both have degree 2

but their sum (i.e. $6+9x$) has degree 1.

What is true, however, is that for each non-negative integer n, the polynomials of degree n or less form a Subspace of $F(-\infty,\infty)$. We will denote this subspace by P_n.

THEOREM: If $W_1, W_2, \dots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.

'SMALLEST SUBSPACE' OF A VECTOR SPACE

If $\mathbf{W}$ is a vector in a vector space V, then $\mathbf{W}$ is said to be a linear combination of the vectors $V_1, V_2, \dots, V_r$ in V if $\mathbf{W}$ can be expressed in the form

$$\mathbf{W} = k_1V_1 + k_2V_2 + \dots + k_rV_r \,, \qquad \text{—— (1)}$$

where $k_1, k_2, \dots, k_r$ are scalars. These scalars are called Coefficients of linear combination.

If $k=1$, then eqn (1) has the form $\quad \mathbf{W} = k_1\mathbf{V}_1 \,, \qquad$ —— (2)

in which case the linear combination is just a scalar multiple of $\mathbf{V}_1$.

THEOREM: If $S = \{\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_r\}$ is a non-empty set of vectors in a vector space V, then (i) The set W of all possible linear combination of vectors in S is a Subspace of V.

(ii) The set W in part (i) is the 'smallest' subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains W.

THE SPAN OF S

The subspace of a vector space V that is formed from all possible linear combinations of the vectors in a non-empty set S is called the Span of S, and we say that the vectors in S span that subspace.

If $S = \{\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_r\}$, then we denote span of S by span$\{\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_r\}$ OR span(S).

Example (11) The Standard Unit Vectors Span R^n

Recall that the standard unit vectors in R^n are

$$\mathbf{e}_1 = (1,0,0,\dots,0)\,,\quad \mathbf{e}_2 = (0,1,0,\dots,0)\,,\ \dots\ ,\mathbf{e}_n = (0,0,0,\dots,1)$$

These vectors span R^n since every vector $\mathbf{V} = (V_1, V_2, \dots, V_n)$ in R^n can be expressed as

$$\mathbf{V} = (V_1, V_2, \dots, V_n) = V_1(1,0,0,\dots,0) + V_2(0,1,0,\dots,0) + \dots + V_n(0,0,0,\dots,1)$$

$$\mathbf{V} = V_1\mathbf{e}_1 + V_2\mathbf{e}_2 + \dots + V_n\mathbf{e}_n \,,$$ which is a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$.

Thus, for example, the vectors $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{k} = (0,0,1)$ span R^3, since every vector $\mathbf{V} = (a,b,c)$ in this space can be expressed as

$$\begin{aligned} \mathbf{V} &= (a,b,c) \\ &= a(1,0,0) + b(0,1,0) + c(0,0,1) \\ &= a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \end{aligned}$$

Example (12) A Geometric View of Spanning in R^2 and R^3

(i) If $\mathbf{v}$ is a non-zero vector in R^2 or R^3 that has its initial point at the origin, then span$\{\mathbf{v}\}$, which is the set of all scalar multiples of $\mathbf{v}$, is the line through the origin determined by $\mathbf{v}$. You should be able to visualize this from fig (i) by observing that the tip of the vector $k\mathbf{v}$ can be made to fall at any point on the line by choosing the value of k appropriately.

(ii) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are non-zero vectors in R^3 that have their initial points at the origin, then span$\{\mathbf{v}_1, \mathbf{v}_2\}$, which consists of all linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$, is the plane through the origin determined by these two vectors. You should be able to visualize this from fig (ii) by observing that the tip of the vector $k_1\mathbf{v}_1 + k_2\mathbf{v}_2$ can be made to fall at any point in the plane by adjusting the scalars k_1 and k_2 to lengthen, shorten, or reverse the directions of the vectors $k_1\mathbf{v}_1$ and $k_2\mathbf{v}_2$ appropriately.

Example (13) A Spanning Set for P_n

The polynomials $1, x, x^2, \cdots, x^n$ span the vector space P_n defined in Example (10) since each polynomial $\mathbf{p}$ in P_n can be written as

$$\mathbf{p} = a_0 + a_1x + \cdots + a_nx^n$$

which is a linear combination of $1, x, x^2, \cdots, x^n$.

We can denote this by writing $P_n = \text{span}\{1, x, x^2, \cdots, x^n\}$.

Example (14) Linear Combinations

Consider the vectors $\mathbf{u} = (1,2,-1)$ and $\mathbf{v} = (6,4,2)$ in R^3. Show that $\mathbf{w} = (9,2,7)$ is a linear combination of $\mathbf{u}$ & $\mathbf{v}$ and that $\mathbf{w}' = (4,-1,8)$ is not a linear combination of $\mathbf{u}$ & $\mathbf{v}$.

Soln.

In order for $\mathbf{w}$ to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$, there must be scalars k_1 & k_2 such that

$$\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v} \quad \text{——— (1)}$$

ie, $(9,2,7) = k_1(1,2,-1) + k_2(6,4,2)$

ie., $(9,2,7) = (k_1+6k_2,\ 2k_1+4k_2,\ -k_1+2k_2)$

Equating corresponding components gives

$$\left.\begin{aligned} k_1+6k_2 &= 9 \\ 2k_1+4k_2 &= 2 \\ -k_1+2k_2 &= 7 \end{aligned}\right\} \quad \text{—— (2)}$$

Augmented matrix for the system (2) is

$$[A|b] = \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 8 & 16 \end{array}\right] \quad R_2 \to \left(-\tfrac{1}{8}\right)R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - 8R_2$$

which yields

$$\begin{aligned} k_1 + 6k_2 &= 9 \\ 0k_1 + k_2 &= 2 \\ 0k_1 + 0k_2 &= 0 \end{aligned}$$

Solving these equations, we get $k_1 = -3$ & $k_2 = 2$

Hence, from (1) $\quad \mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$

Similarly, for $\mathbf{w}'$ to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$, there must be scalars k_1 and k_2 such that
$$\mathbf{w}' = k_1\mathbf{u} + k_2\mathbf{v}$$

$$\text{ie;}\quad (4,-1,8) = k_1(1,2,-1) + k_2(6,4,2)$$

$$\Rightarrow \quad (4,-1,8) = (k_1+6k_2,\ 2k_1+4k_2,\ -k_1+2k_2)$$

Equating corresponding components give

$$\left.\begin{aligned} k_1+6k_2 &= 4 \\ 2k_1+4k_2 &= -1 \\ -k_1+2k_2 &= 8 \end{aligned}\right\} \quad - ③$$

Augmented matrix for ③ is

$$[A|b] = \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & 1 & \frac{9}{8} \\ 0 & 8 & 12 \end{array}\right] \quad R_2 \to \left(-\tfrac{1}{8}\right)R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & 1 & \frac{9}{8} \\ 0 & 0 & 3 \end{array}\right] \quad R_3 \to R_3 + 8R_2$$

which yields

$$\begin{aligned} k_1 + 6k_2 &= 4 \\ 0k_1 + k_2 &= \frac{9}{8} \\ 0k_1 + 0k_2 &= 3 \end{aligned}$$

No value of k_1 and k_2 satisfy the last equ. $(0k_1 + 0k_2 = 3)$ so the system of equ. ③ is Inconsistent.

Consequently, $\mathbf{w}'$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Example (15) Testing for Spanning

Determine whether $\mathbf{v}_1 = (1,1,2)$, $\mathbf{v}_2 = (1,0,1)$ and $\mathbf{v}_3 = (2,1,3)$ span the vector space $\mathbf{R}^3$.

Solution: We must determine whether an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ in R^3 can be expressed as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$

i.e., $$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 \quad \text{——— (1)}$$

or $$(b_1, b_2, b_3) = k_1(1,1,2) + k_2(1,0,1) + k_3(2,1,3)$$

or $$(b_1, b_2, b_3) = (k_1 + k_2 + 2k_3,\ k_1 + k_3,\ 2k_1 + k_2 + 3k_3)$$

Equating corresponding components,

$$\left.\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 \quad + k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned}\right\} \quad \text{——— (2)}$$

Thus, our problem reduces to ascertaining whether this system of eqn. is Consistent for all values of b_1, b_2 and b_3.

One way of doing this is to use the theorem which state that the system of eqn. is Consistent iff its coefficient matrix has a non-zero determinant.

For system of eqn. (2), $$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= 1(0-1) - 1(1-0) + 2(1-0)$$

$$= -1 - 1 + 2$$

$$= 0$$

Hence system of eqn. (2) is Inconsistent so $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ do not span R^3.

SEC (4.3) LINEAR INDEPENDENCE

In this Section, we will consider the question of whether the vectors in a given set are interrelated in the sense that one or more of them can be expressed as a linear combination of others.

Definition If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$ is a non-empty set of vectors in a vector space V, then the vector eqn.

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

has at least one solution, namely $k_1 = 0,\ k_2 = 0,\ \cdots,\ k_r = 0$

If this is the only solution (called trivial soln.), then S is said to be Linearly Independent.
If there are solutions in addition to trivial soln., then S is said to be Linearly Dependent Set.

NOTE We will often apply the terms Linearly Independent and Linearly Dependent to the vectors themselves rather than to the set.

Example ① Linear Independence of the Standard Unit Vectors in R^n

The most basic linearly independent set in R^n is the set of standard unit vectors

$$\mathbf{e}_1 = (1,0,0,\cdots,0),\ \mathbf{e}_2 = (0,1,0,\cdots,0),\ \cdots,\ \mathbf{e}_n = (0,0,0,\cdots,1)$$

for notational simplicity, we will prove that the standard unit vectors in R^3 i.e., $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$ are linearly independent.

Let $k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k} = \mathbf{0}$ ——— ①

$$\Rightarrow k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) = (0,0,0)$$

$$\Rightarrow (k_1, k_2, k_3) = (0,0,0)$$

$$\Rightarrow k_1 = 0,\ k_2 = 0,\ k_3 = 0$$

This implies that ① has only trivial soln. and hence the vectors are linearly independent.

Example ② Linear Independence in R^3

Determine whether the vectors $\mathbf{v}_1 = (1,-2,3)$, $\mathbf{v}_2 = (5,6,-1)$, $\mathbf{v}_3 = (3,2,1)$ are linearly independent or linearly dependent in R^3.

Soln. The linear independence or linear dependence of these vectors are determined by whether there exist non-trivial solutions of the vector eqn.

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0} \quad \text{——— ①}$$

or $$k_1(1,-2,3) + k_2(5,6,-1) + k_3(3,2,1) = (0,0,0)$$

or $$(k_1+5k_2+3k_3,\ -2k_1+6k_2+2k_3,\ 3k_1-k_2+k_3) = (0,0,0)$$

Equating corresponding components yields the homogeneous linear system

$$\left.\begin{aligned} k_1+5k_2+3k_3 &= 0 \\ -2k_1+6k_2+2k_3 &= 0 \\ 3k_1-k_2+k_3 &= 0 \end{aligned}\right\} \quad \text{——— ②}$$

Thus, our problem reduces to determining whether the system ② has non-trivial solutions.
There are various ways to do this —

Method-(I) First method is to simply solve the system (by Gauss-Elimination)

Augmented matrix, $[A|b] = \left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & -16 & -8 & 0 \end{array}\right]$ $\quad R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 3R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right]$ $\quad R_2 \to \frac{1}{16} R_2$, $R_3 \to \left(-\frac{1}{16}\right) R_3$

$\sim \begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad R_3 \to R_3 - R_2$

which yields
$$k_1 + 5k_2 + 3k_3 = 0 \quad \text{—— (i)}$$
$$k_2 + \tfrac{1}{2} k_3 = 0 \quad \text{—— (ii)}$$

Let $k_3 = t$, then from (ii), $k_2 + \frac{1}{2} t = 0$ i.e., $k_2 = -\frac{t}{2}$

and from (i), $k_1 + 5\left(-\frac{t}{2}\right) + 3t = 0$ i.e., $k_1 = -\frac{t}{2}$

Thus, the system ② has non-trivial solutions and hence the vectors are linearly dependent.

Method-(II) (When Coefficient matrix is Square Matrix)

A second method for obtaining the same result is to compute the determinant of coefficient matrix i.e.,

$$|A| = \begin{vmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{vmatrix} = 1\,(6+2) - 5\,(-2-6) + 3\,(2-18)$$
$$= 8 + 40 - 48$$
$$= 0$$

Thus, the system ③ has non-trivial solutions and the vectors are linearly dependent.

NOTE — If $|A| = 0$, then vectors are linearly dependent and
if $|A| \neq 0$, then vectors are linearly independent.

Example ③ Linear Independence in R^4

Determine whether the vectors $\mathbf{v}_1 = (1,2,2,-1)$, $\mathbf{v}_2 = (4,9,9,-4)$, $\mathbf{v}_3 = (5,8,9,-5)$ in R^4 are linearly dependent or linearly independent.

Solution.

Let $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ ——— ①

$$\Rightarrow k_1(1,2,2,-1) + k_2(4,9,9,-4) + k_3(5,8,9,-5) = (0,0,0,0)$$

$$\Rightarrow (k_1+4k_2+5k_3,\ 2k_1+9k_2+8k_3,\ 2k_1+9k_2+9k_3,\ -k_1-4k_2-5k_3) = (0,0,0,0)$$

Equating corresponding components yields the homogeneous linear system

$$\left.\begin{aligned} k_1+4k_2+5k_3 &= 0 \\ 2k_1+9k_2+8k_3 &= 0 \\ 2k_1+9k_2+9k_3 &= 0 \\ -k_1-4k_2-5k_3 &= 0 \end{aligned}\right\} \quad \text{—— ②}$$

Augmented matrix for the system ② is

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 2 & 9 & 8 & 0 \\ 2 & 9 & 9 & 0 \\ -1 & -4 & -5 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 + R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc} 1 & 4 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$

which yields

$$k_1 + 4k_2 + 5k_3 = 0 \quad \text{—— (i)}$$
$$k_2 - 2k_3 = 0 \quad \text{—— (ii)}$$
$$k_3 = 0 \quad \text{—— (iii)}$$

Solving these equations, we get $k_3 = 0$, $k_2 = 0$, $k_1 = 0$

Thus, the system ② has only trivial solution and hence the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

Example (4) An Important Linearly Independent Set in P_n

Show that the polynomials $1, x, x^2, \ldots, x^n$ form a linearly independent set in P_n.

Solution. Let us denote the polynomials as

$$\mathbf{p}_0 = 1,\ \mathbf{p}_1 = x,\ \mathbf{p}_2 = x^2,\ \ldots\ldots,\ \mathbf{p}_n = x^n$$

The linear independence or linear dependence of these vectors is determined by whether there exist non-trivial solutions of the vector eqn.

$$a_0\mathbf{p}_0 + a_1\mathbf{p}_1 + a_2\mathbf{p}_2 + \ldots\ldots + a_n\mathbf{p}_n = \mathbf{0} \qquad \text{—— (1)}$$

i.e., $a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n = \mathbf{0}$, for all x in $(-\infty, \infty)$

This holds iff $a_0 = a_1 = a_2 = \ldots\ldots = a_n = 0$

Thus (1) has only trivial soln. so the vectors $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are linearly independent.

Example (5) Linear Independence of Polynomials

Determine whether the polynomials $\mathbf{p}_1 = 1 - x$, $\mathbf{p}_2 = 5 + 3x - 2x^2$, $\mathbf{p}_3 = 1 + 3x - x^2$ are linearly dependent or linearly independent in P_2.

Solution. Let $k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 = \mathbf{0}$ —— (1)

i.e., $k_1(1-x) + k_2(5+3x-2x^2) + k_3(1+3x-x^2) = \mathbf{0}$

$$\Rightarrow (k_1 + 5k_2 + k_3) + (-k_1 + 3k_2 + 3k_3)x + (-2k_2 - k_3)x^2 = \mathbf{0}$$

Since this eqn. must be satisfied by all x in $(-\infty, \infty)$, each coefficient must be zero

i.e.,
$$\left.\begin{aligned} k_1 + 5k_2 + k_3 &= 0 \\ -k_1 + 3k_2 + 3k_3 &= 0 \\ -2k_2 - k_3 &= 0 \end{aligned}\right\} \qquad \text{—— (2)}$$

Coefficient matrix of (2) is
$$A = \begin{bmatrix} 1 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{bmatrix}$$

Now
$$|A| = 1(-3+6) - 5(1-0) + 1(2-0)$$
$$= 3 - 5 + 2$$
$$= 0$$

Thus the linear system (2) has non-trivial solutions and so the polynomials $\mathbf{p}_1, \mathbf{p}_2$ & $\mathbf{p}_3$ are linearly dependent.

An Alternative Interpretation of Linear Independence :-

The terms Linearly Dependent and Linearly Independent are intended to indicate whether the vectors in a given set are interrelated in some way. The following theorem makes this idea more precise.

THEOREM: A set S with two or more vectors is

(i) Linearly Dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

(ii) Linearly Independent if and only if no vector in S is expressible as a linear combination of the other vectors in S.

Example ⑥ Example ② Revisited

In Example ②, we saw that the vectors

$\mathbf{v}_1 = (1,-2,3)$, $\mathbf{v}_2 = (5,6,-1)$ & $\mathbf{v}_3 = (3,2,1)$ are linearly dependent.

To find Relation between $\mathbf{v}_1, \mathbf{v}_2$ & $\mathbf{v}_3$

putting the values of k_1, k_2 & k_3 (determined by Method-I) in eqn. ①, we get

$$-\frac{t}{2}\mathbf{v}_1 - \frac{t}{2}\mathbf{v}_2 + t\,\mathbf{v}_3 = 0$$

$$\Rightarrow \quad -t\left(\frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 - \mathbf{v}_3\right) = 0$$

$$\Rightarrow \quad \frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 - \mathbf{v}_3 = 0$$

or $\quad \mathbf{v}_3 = \frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2$, which confirms Theorem (i).

SETS WITH ONE OR TWO VECTORS :- The following basic theorem is concerned with the linear independence and linear dependence of sets with one or two vectors and the sets that contain the zero vector.

THEOREM: (i) A finite set that contains $\mathbf{0}$ is linearly dependent.

(ii) A set with exactly one vector is linearly independent iff that vector is not $\mathbf{0}$.

(iii) A set with exactly two vectors is linearly independent iff neither vector is a scalar multiple of the other.

Example ⑦ Linear Independence of Two functions

The functions $\mathbf{f}_1 = x$ and $\mathbf{f}_2 = \sin x$ are linearly independent vectors in $F(-\infty,\infty)$ since neither function is a scalar multiple of the other.

On the other hand, the two functions $\mathbf{g}_1 = \sin 2x$ and $\mathbf{g}_2 = \sin x \cos x$ are linearly dependent because the trigonometric identity $\sin 2x = 2\sin x \cdot \cos x$ reveals that $\mathbf{g}_1$ and $\mathbf{g}_2$ are scalar multiples of the other.

THEOREM. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$ be a set of vectors in R^n.
If $r > n$, then the set S is linearly dependent.

NOTE: It follows from this theorem that a set in R^2 with more than two vectors is linearly dependent and a set in R^3 with more than three vectors is linearly independent

LINEAR INDEPENDENCE OF FUNCTIONS —

There is no general method that can be used to determine whether a set of functions is linearly independent or linearly dependent. However, there does exist a theorem that is useful for establishing linear independence in certain circumstances. The following definition will be useful for discussing that theorem.

Definition: If $\mathbf{f}_1 = f_1(x)$, $\mathbf{f}_2 = f_2(x)$, $\cdots$, $\mathbf{f}_n = f_n(x)$ are functions that are n-1 times differentiable on the interval $(-\infty,\infty)$, then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{n-1}(x) & f_2^{n-1}(x) & \cdots & f_n^{n-1}(x) \end{vmatrix}$$

is called the Wronskian of $\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n$.

THEOREM: If the functions $\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n$ have n-1 continuous derivatives on the interval $(-\infty,\infty)$ and if the Wronskian of these functions is not identically zero on $(-\infty,\infty)$, then these functions form a linearly independent set of vectors in $C^{n-1}(-\infty,\infty)$.

NOTE. The Converse of this Theorem is false. If the Wronskian of $\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n$ is identically zero on $(-\infty,\infty)$, then no conclusion can be reached about the linear independence of $\{\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n\}$ — this set of vectors may be linearly independent or dependent

Example ⑧ **Linear Independence Using Wronskian**

Use the Wronskian to show that $f_1 = x$ and $f_2 = \sin x$ are linearly independent.

Solution. The Wronskian of f_1 & f_2 is

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix}$$

$$= x\cos x - \sin x$$

This function is not identically zero on the interval $(-\infty, \infty)$ since for example,

$$W\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{2}\cos\tfrac{\pi}{2} - \sin\tfrac{\pi}{2}$$

$$= \tfrac{\pi}{2}(0) - 1 = -1$$

Thus, the functions are linearly independent.

Example ⑨ Use the Wronskian to show that $f_1 = 1$, $f_2 = e^x$ and $f_3 = e^{2x}$ are linearly independent.

Soln. The Wronskian of f_1, f_2 & f_3 is

$$W(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$= \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix}$$

$$= 1\,(4e^{3x} - 2e^{3x}) \text{, expanding by first column}$$

$$= 2e^{3x}$$

This function is obviously not identically zero on $(-\infty, \infty)$, so f_1, f_2 and f_3 form a linearly independent set.

SEC (4.4) CO-ORDINATES AND BASIS.

We usually think of a line as being one-dimensional, a plane as two-dimensional and the space around us as three-dimensional. In this Section, we will discuss coordinates systems in general vector spaces and lay the groundwork for a precise definition of Dimension in the next Section.

Coordinate Systems in Linear Algebra — from Text Book

BASIS FOR A VECTOR SPACE — The following definition will enable us to extend the concept of a co-ordinate system to general vector spaces.

Definition: If V is any vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a finite set of vectors in V, then S is called a Basis for V if the following two conditions hold—

(i) The set S is Linearly Independent. [The vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent]

(ii) The set S spans V. [Every vector in V can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$]

NOTE. If we think of a Basis as describing a co-ordinate system for a vector space in V, then part (i) of this definition guarantees that there is no interrelationship between the basis vectors and part (ii) guarantees that there are enough basis vector to provide co-ordinates for all vectors in V.

Example ① The Standard Basis for R^n

Show that $S = \{e_1, e_2, \cdots, e_n\}$ is a basis for R^n, where $e_1 = (1,0,0,\cdots,0)$, $e_2 = (0,1,0,\cdots,0)$ etc.

Soln. We must show that the vectors in S are linearly independent and span R^n.

(i) To show that S is linearly independent:

$$\text{let} \quad k_1 e_1 + k_2 e_2 + \cdots + k_n e_n = \mathbf{0}$$

$$\Rightarrow \quad k_1(1,0,0,\cdots,0) + k_2(0,1,0,\cdots,0) + \cdots\cdots + k_n(0,0,0,\cdots,1) = (0,0,0,\cdots,0)$$

$$\Rightarrow \quad (k_1, k_2, \cdots, k_n) = (0,0,0,\cdots,0)$$

$$\Rightarrow \quad k_1 = 0,\ k_2 = 0,\ \cdots,\ k_n = 0$$

$\Rightarrow$ Vectors in S are linearly independent.

(ii) To show that S spans R^n: Every vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ in R^n can be expressed as

$$\mathbf{v} = (v_1, v_2, \cdots, v_n) = v_1(1,0,0,\cdots,0) + v_2(0,1,0,\cdots,0) + \cdots + v_n(0,0,\cdots,1)$$
$$= v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$$

$\Rightarrow$ The vectors $e_1, e_2, \cdots, e_n$ span R^n

Thus these vectors form a Basis for R^n that we call the Standard Basis for R^n.

NOTE: In particular $S = \{\hat{i}, \hat{j}\}$ is the Standard Basis for R^2,
where $\hat{i} = (1,0)$ and $\hat{j} = (0,1)$
and $S = \{\hat{i}, \hat{j}, \hat{k}\}$ is the Standard Basis for R^3,
where $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$ and $\hat{k} = (0,0,1)$

Example ② The Standard Basis for P_n

Show that $S = \{1, x, x^2, \cdots, x^n\}$ is a Basis for the vector space P_n of polynomials of degree n or less.

Solution. Let us denote these polynomials by

$$p_0 = 1, \quad p_1 = x, \quad p_2 = x^2, \quad \cdots\cdots, \quad p_n = x^n$$

We must show that $p_0, p_1, p_2, \cdots, p_n$ are linearly independent and span P_n.

(i) To show that $p_0, p_1, p_2, \cdots, p_n$ are linearly independent

Let $k_0 p_0 + k_1 p_1 + k_2 p_2 + \cdots + k_n p_n = 0$

$$\Rightarrow k_0 + k_1 x + k_2 x^2 + \cdots\cdots + k_n x^n = 0, \quad \text{for all } x \in (-\infty, \infty)$$

This holds iff $k_0 = k_1 = k_2 = \cdots = k_n = 0$

so the vectors $p_0, p_1, p_2, \cdots, p_n$ are linearly independent.

(ii) To show that $\{p_0, p_1, p_2, \cdots, p_n\}$ span P_n

Each polynomial p in P_n can be written as

$$p = k_0 + k_1 x + k_2 x^2 + \cdots + k_n x^n$$

$$= k_0 p_0 + k_1 p_1 + k_2 p_2 + \cdots + k_n p_n$$

which is a linear combination of $p_0, p_1, \cdots\cdots, p_n$.

$\Rightarrow$ These vectors span P_n.

Thus, the vectors $p_0, p_1, p_2, \cdots, p_n$ form a Basis for P_n that we call Standard Basis for P_n.

NOTE: In particular, the standard basis for P_2 is $\{1, x, x^2\}$.
and the standard basis for P_3 is $\{1, x, x^2, x^3\}$.

Example ③ Another Basis for R^3

Show that the vectors $V_1 = (1,2,1)$, $V_2 = (2,9,0)$ and $V_3 = (3,3,4)$ form a Basis for R^3.

Soln. We must show that these vectors are linearly independent and span R^3.

(i) To show that vectors are linearly independent

$$\text{Let} \quad k_1V_1 + k_2V_2 + k_3V_3 = \mathbf{0} \qquad \text{—— ①}$$

$$\text{i.e.,} \quad k_1(1,2,1) + k_2(2,9,0) + k_3(3,3,4) = (0,0,0)$$

$$\Rightarrow (k_1+2k_2+3k_3,\ 2k_1+9k_2+3k_3,\ k_1+4k_3) = (0,0,0)$$

$$\Rightarrow \left.\begin{aligned} k_1+2k_2+3k_3 &= 0 \\ 2k_1+9k_2+3k_3 &= 0 \\ k_1 \qquad + 4k_3 &= 0 \end{aligned}\right\} \qquad \text{—— ②}$$

$$\text{Now} \quad |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix}$$

$$= 1\begin{vmatrix} 9 & 3 \\ 0 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & 9 \\ 1 & 0 \end{vmatrix}$$

$$= 1(36-0) - 2(8-3) + 3(0-9)$$

$$= -1$$

Thus, the homogeneous system ② has only trivial soln. i.e., $k_1 = 0$, $k_2 = 0$, $k_3 = 0$

so the vector V_1, V_2 and V_3 are Linearly Independent.

(ii) To show that vectors span R^3

We must show that every vector $\mathbf{b} = (b_1, b_2, b_3)$ in R^3 can be expressed as

$$k_1V_1 + k_2V_2 + k_3V_3 = \mathbf{b} \qquad \text{—— ③}$$

$$\text{i.e.,} \quad k_1(1,2,1) + k_2(2,9,0) + k_3(3,3,4) = (b_1, b_2, b_3)$$

$$\Rightarrow \left.\begin{aligned} k_1+2k_2+3k_3 &= b_1 \\ 2k_1+9k_2+3k_3 &= b_2 \\ k_1 \qquad + 4k_3 &= b_3 \end{aligned}\right\} \qquad \text{—— ④}$$

$$\text{Now Again} \quad |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = -1 \quad \text{(As before)}$$

Thus, the non-homogeneous system is consistent for all values of b_1, b_2 and b_3.

$\Rightarrow$ The vectors V_1, V_2, V_3 span R^3

Hence, the vectors V_1, V_2, V_3 form a Basis for R^3.

COORDINATES RELATIVE TO A BASIS

THEOREM Uniqueness of Basis Representation

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a Basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ in exactly one way.

Definition. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for a vector space V, and

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

is the expression for a vecter $\mathbf{v}$ in terms of the basis S, then the scalars $c_1, c_2, \cdots, c_n$ are called the Co-ordinates of $\mathbf{v}$ relative to the basis S. The vector $(c_1, c_2, \cdots, c_n)$ in R^n constructed from these co-ordinates is called the Coordinate Vector of $\mathbf{v}$ relative to S and it is denoted by

$$(\mathbf{v})_S = (c_1, c_2, \cdots, c_n)$$

NOTE. Sometimes it is desirable to write a Co-ordinate vector as a column matrix, in which case, we denote it using square brackets as

$$[\mathbf{v}]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

We will refer to $[\mathbf{v}]_S$ as a Co-ordinate Matrix and reserve the terminology Co-ordinate Vector for the comma delimited form $(\mathbf{v})_S$.

NOTE Recall that two sets are considered to be the same if they have the same members even if those members are written in a different order. However, if $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a set of basis vectors, then changing the order in which the vectors are written would change the order of the entries in $(\mathbf{v})_S$, possibly producing a different co-ordinate vector. To avoid this complication, we will make the convention that in any discussion involving a basis S, the order of the vectors in S remains fixed. Some authors call a set of basis vectors with this restriction an Ordered Basis.

Example Co-ordinates Relative to the Standard Basis for R^n

In the special case where $V = R^n$ and S is the Standard Basis, the co-ordinate vector $(\mathbf{v})_S$ and the vector $\mathbf{v}$ are the same; that is, $\mathbf{v} = (\mathbf{v})_S$

For example, in R^3 the representation of a vector $\mathbf{v} = (a, b, c)$ as a linear combination of the vectors in the standard basis $S = \{\hat{i}, \hat{j}, \hat{k}\}$ is

$$\mathbf{v} = a\hat{i} + b\hat{j} + c\hat{k}$$

so the co-ordinate vector relative to this basis is $(\mathbf{v})_S = (a, b, c)$ which is the same as the vector $\mathbf{v}$.

Example **Co-ordinate Vectors Relative to Standard Bases**

(i) find the coordinate vector for the polynomial $p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$ relative to the standard basis for the vector space P_n.

(ii) find the coordinate vector of $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ relative to standard basis for M_{22}.

Soln. (i) The given formula for $p(x)$ expresses this polynomial as a linear combination of the standard basis vector $S = \{1, x, x^2, \cdots, x^n\}$.

Thus, the co-ordinate vector for p relative to S is $(p)_S = (c_0, c_1, c_2, \cdots, c_n)$.

(ii) The representation of a vector $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as a linear combination of the standard basis vector is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so the coordinate vector of B relative to S is

$$(B)_S = (a, b, c, d)$$

Example **Co-ordinates in R^n**

(i) We showed in Example ③ that the vectors $v_1 = (1,2,1)$, $v_2 = (2,9,0)$ & $v_3 = (3,3,4)$ form a Basis for R^3.

Find the co-ordinate vector of $v = (5,-1,9)$ relative to the Basis $S = (v_1, v_2, v_3)$.

(ii) find the vectors in R^3 whose co-ordinate vector relative to S is $(v)_S = (-1,3,2)$.

Soln. (i) To find $(v)_S$, we must find values of c_1, c_2 and c_3 such that

$$v = c_1v_1 + c_2v_2 + c_3v_3 \quad \text{—— ①}$$

ie.,
$$(5,-1,9) = c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4)$$

$$\Rightarrow (c_1 + 2c_2 + 3c_3,\ 2c_1 + 9c_2 + 3c_3,\ c_1 + 4c_3) = (5,-1,9)$$

Equating corresponding components gives

$$\left.\begin{aligned} c_1 + 2c_2 + 3c_3 &= 5 \\ 2c_1 + 9c_2 + 3c_3 &= -1 \\ c_1 \qquad + 4c_3 &= 9 \end{aligned}\right\} \quad \text{—— ②}$$

Solving the system of eqn. ②, we get $c_1 = 1$, $c_2 = -1$, $c_3 = 2$

Therefore $(v)_S = (1,-1,2)$.

(ii) Using the defi. of $(v)_S$, we obtain

$$\begin{aligned} v &= (-1)v_1 + 3v_2 + 2v_3 \\ &= (-1)(1,2,1) + 3(2,9,0) + 2(3,3,4) \\ &= (-1,-2,-1) + (6,27,0) + (6,6,8) \\ &= (-1+6+6,\ -2+27+6,\ -1+0+8) = (11,31,7) \end{aligned}$$

SEC (4.5) DIMENSION

We showed in the previous Section that the standard basis for R^n has n vectors and hence that the standard basis for R^3 has three vectors, the standard basis for R^2 has two vectors and the standard basis for $R^1 (= R)$ has one vector.

Since we think of space as three dimensional, a plane as two dimensional and a line as one dimensional, there seems to be a link between the no. of vectors in a basis and the dimension of a vector space. We will develop this idea in this Section.

Number of Vectors in a Basis

THEOREM (1) All bases for a finite-dimensional vector space have the same no. of vectors.

THEOREM (2) Let V is a finite-dimensional vector space and let $\{v_1, v_2, \cdots, v_n\}$ is any basis

(i) If a set has more than n vectors, then it is linearly dependent.

(ii) If a set has fewer than n vectors, then it does not span V.

We can now see rather easily why Theorem (1) is true; for if $S = \{v_1, v_2, \cdots, v_n\}$ is an arbitrary basis for V, then the linear independence of S implies that any set in V with more than n vectors is linearly dependent and any set in V with fewer than n vectors does not span V. Thus, unless a set in V has exactly n vectors, it cannot be a basis.

Definition: The Dimension of a finite-dimensional vector space V is defined to be the no. of vectors in a basis for V and is denoted by $\dim(V)$.

In addition, the zero vector space is defined to have dimension zero.

NOTE. Some writers regard the empty set to be a basis for the zero vector space. This is consistent with our defi. of dimension, since the empty set has no vectors and the zero vector space has dimension zero.

Example (1) Dimensions of Some familiar Vector Spaces

$\dim(R^n) = n$ The standard basis has n vectors.

$\dim(P_n) = n+1$ The standard basis has $n+1$ vectors.

$\dim(M_{mn}) = mn$ The standard basis has mn vectors.

Example (2) Dimension of Span(S)

If $S = \{v_1, v_2, \cdots, v_r\}$ is a linearly independent set in a vector space V, then S is automatically a basis for span(S) and this implies that $\dim[\text{span}(S)] = r$.

In words, the dimension of the space spanned by a linearly independent set of vectors is equal to the no. of vectors in that set.

Example ③ Dimension of a Solution Space

Find a basis for and the dimension of the solu. space of the homogeneous system –

$$
\begin{aligned}
2x_1+2x_2-x_3 \quad\quad +x_5 &= 0\\
-x_1-x_2+2x_3-3x_4+x_5 &= 0\\
x_1+x_2-2x_3 \quad\quad -x_5 &= 0\\
x_3+x_4+x_5 &= 0
\end{aligned}
$$

Solu. The Augmented matrix for the system is

$$
\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0\\ -1 & -1 & 2 & -3 & 1 & 0\\ 1 & 1 & -2 & 0 & -1 & 0\\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0\\ -1 & -1 & 2 & -3 & 1 & 0\\ 2 & 2 & -1 & 0 & 1 & 0\\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3
$$

$$
\sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0\\ 0 & 0 & 0 & -3 & 0 & 0\\ 0 & 0 & 3 & 0 & 3 & 0\\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}
$$

$$
\sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0\\ 0 & 0 & 1 & 1 & 1 & 0\\ 0 & 0 & 3 & 0 & 3 & 0\\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_4
$$

$$
\sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0\\ 0 & 0 & 1 & 1 & 1 & 0\\ 0 & 0 & 0 & -3 & 0 & 0\\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 3R_2
$$

$$
\sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0\\ 0 & 0 & 1 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix} \quad R_3 \to \left(-\tfrac{1}{3}\right) R_3
$$

$$
\sim \begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0\\ 0 & 0 & 1 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 + 3R_3
$$

The corresponding system of equations is

$$
\begin{aligned}
x_1+x_2-2x_3 \quad -x_5 &= 0\\
x_3+x_4+x_5 &= 0\\
x_4 &= 0
\end{aligned}
$$

Solving for the leading variables, we obtain

$$\left.\begin{aligned} x_1 &= -x_2 + 2x_3 + x_5 \\ x_3 &= -x_4 - x_5 \\ x_4 &= 0 \end{aligned}\right\}$$

If we now assign the free variables x_2 & x_5 arbitrary values s & t resp., then we can express the solution set as

$$x_1 = -s-t\ ,\ x_2 = s\ ,\ x_3 = -t\ ,\ x_4 = 0\ ,\ x_5 = t$$

which can be written in vector form as

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) &= (-s-t, s, -t, 0, t) \\ &= (-s-t,\ s+0t,\ 0s-t,\ 0s+0t,\ 0s+t) \\ &= s(-1,1,0,0,0) + t(-1,0,-1,0,1)\end{aligned}$$

This shows that the vectors $\mathbf{v}_1 = (-1,1,0,0,0)$ and $\mathbf{v}_2 = (-1,0,-1,0,1)$ span the solution space. Since neither vector is a scalar multiple of the other, they are linearly independent and hence form a basis for the solution space.

Thus, the solution space of the system has dimension 2.

<u>Some Fundamental Theorems</u>: We will start with a theorem that is concerned with the effect on linear independence and spanning if a vector is added to or removed from a given non-empty set of vectors. Informally stated, if you start with a linearly independent set S and adjoin to it a vector that is not a linear combination of those in S, then the enlarged set will still be linearly independent. Also, if you start with a set S of two or more vectors in which one of the vectors is a linear combination of the others, then that vector can be removed from S without affecting span(S).

<u>THEOREM</u> <u>Plus/Minus Theorem</u>

Let S be a non-empty set of vectors in a vector spac V.

(i) If S is a linearly independent set and if $\mathbf{v}$ is a vector in V that is outside of span(S), then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into S is still linearly independent.

(ii) If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then $S - \{\mathbf{v}\}$ span the same space; that is

$$\text{span}(S) = \text{span}(S - (\mathbf{v}))$$