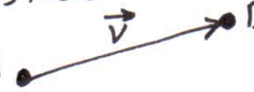
SEC 3.1 VECTORS IN 2-SPACE, 3-SPACE, and n-SPACE

In this Section, we will introduce some of the basic ideas about vectors. As we progress through the text, we will see that Vectors and Matrices are closely related and that much of Linear Algebra is concerned with that relationship.

GEOMETRIC VECTORS: Engineers and physicists represent vectors in two dimensions (also called 2-space) or in three dimensions (also called 3-space) by arrows. The direction of the arrowhead specifies the Direction of the vector and the Length of the arrow specifies the magnitude of the vector. Mathematicians call these Geometric Vectors. The tail of the arrow is called the Initial Point of the Vector and the tip is called Terminal Point.

We will denote vectors in boldface type such as $\vec{a}, \vec{b}, \vec{v}, \vec{W}$ and $\vec{X}$ and we will denote scalars in lowercase italic type such as a, k, v, w, and x.

If a vector $\vec{v}$ has initial point A and terminal point B, then we will write $\vec{V} = \overrightarrow{AB}$



Vectors with the same length and direction are said to be Equivalent. Equivalent Vectors are regarded to be the same vector even though they may be in different positions (See fig.)

Fig. Equivalent Vectors.

The vector whose initial and terminal points coincide has length zero, so we call this the Zero Vector and denote it by $\vec{0}$. The zero vector has no natural direction, so we will agree that it can be assigned any direction that is convenient for problem.

VECTOR ADDITION: There are a no. of important algebraic operations on vectors, all of which have their origin in laws of physics.

Parallelogram Rule for Vector Addition

If $\vec{v}$ and $\vec{w}$ are vectors in 2-space or 3-space that are positioned so their initial points coincide, then the two vectors form adjacent sides of a parallelogram and the sum $\vec{v}+\vec{w}$ is the vector represented by the arrow from the common initial point of $\vec{v}$ and $\vec{w}$ to the opposite vertex of the parallelogram (fig (i))

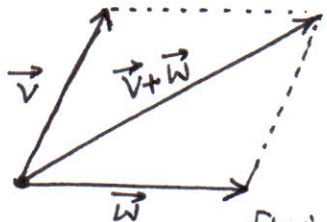
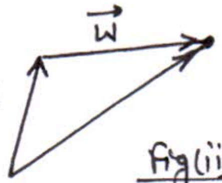


Fig (i)

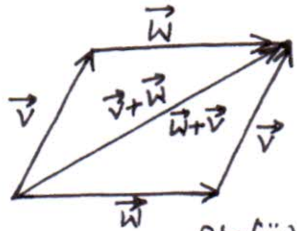
Here is another way to form the sum of two vectors.



Fig(ii)

Triangle Rule for Vector Addition –

If $\vec{v}$ and $\vec{w}$ are vectors in 2-space or 3-space that are positioned so the initial point of $\vec{w}$ is at the terminal point of $\vec{v}$, then the sum $\vec{v}+\vec{w}$ is represented by the arrow from the initial point of $\vec{v}$ to the terminal point of $\vec{w}$. (See fig (ii))



Fig(iii)

NOTE: In fig.(iii), we have constructed the sums $\vec{v}+\vec{w}$ and $\vec{w}+\vec{v}$ by the Triangle rule. This construction makes it evident that

$$\vec{v}+\vec{w} = \vec{w}+\vec{v} \qquad \text{——} \ (1)$$

and that the sum obtained by Triangle Rule is same as the sum obtained by Parallelogram rule.

NOTE: Vector Addition $\vec{v}+\vec{w}$ can be viewed as the Translation of $\vec{v}$ by $\vec{w}$ or, alternatively, the Translation of $\vec{w}$ by $\vec{v}$.

VECTOR SUBTRACTION: The Negative of a vector $\vec{v}$ is the vector that has same length as $\vec{v}$ but is oppositely directed and is denoted by $-\vec{v}$. (See Fig (i))

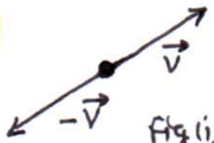


Fig (i)

The Difference of $\vec{v}$ from $\vec{w}$ is taken to be the sum

$$\vec{w}-\vec{v} = \vec{w}+(-\vec{v}) \qquad \text{——} \ (2)$$

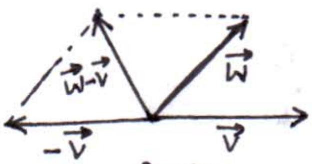


Fig (ii)

The Difference of $\vec{v}$ from $\vec{w}$ can be obtained geometrically by the parallelogram method shown in Fig (ii) OR more directly by positioning $\vec{w}$ and $\vec{v}$ so their initial points coincide and drawing the vector from the terminal point of $\vec{v}$ to the terminal point of $\vec{w}$ (See fig (iii)).

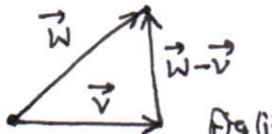
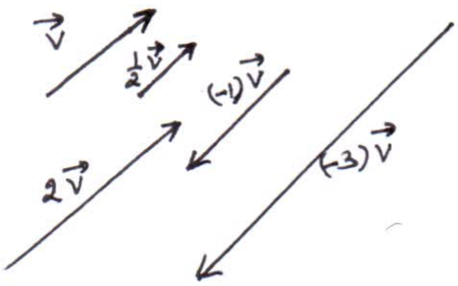


Fig (iii)

SCALAR MULTIPLICATION: If $\vec{v}$ is a non-zero vector in 2-space or 3-space and if k is a non-zero scalar, then we define the Scalar Product of $\vec{v}$ by k to be the vector whose length is $|k|$ times the length of $\vec{v}$ and whose direction is the same as that of $\vec{v}$ if k is +ve and opposite to that if k is -ve. If $k=0$ or $\vec{v}=\vec{0}$, then we define $k\vec{v}$ to be $\vec{0}$

The adjacent fig. shows the geometric relationship between a vector $\vec{v}$ and some of its scalar multiples. In particular, observe that $(-1)\vec{v}$ has same length as $\vec{v}$ but is oppositely directed; therefore,



$$(-1)\vec{v} = -\vec{v} \qquad \text{——} \ (3)$$

SUMS OF THREE OR MORE VECTORS.

Vector Addition satisfies the Associative Law for Addition, that is,

$$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$

A simple way to construct $\vec{u}+\vec{v}+\vec{w}$ is to place the vectors 'tip to tail' in succession and then draw the vector from the initial point of $\vec{u}$ to the terminal point of W (See fig(i)). The 'tip to tail' method also works for four or more vectors (See fig(ii)). The 'tip to tail' method also makes it evident that if $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in 3-space with a common initial point, then $\vec{u}+\vec{v}+\vec{w}$ is the diagonal of the parallelopiped that has three vectors as adjacent sides (See fig(iii)).

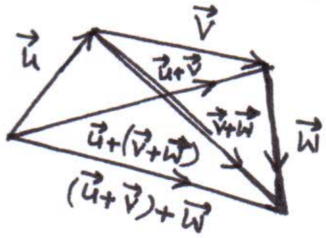


fig (i)

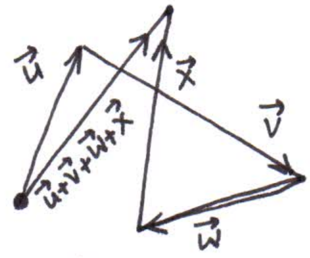


fig (ii)

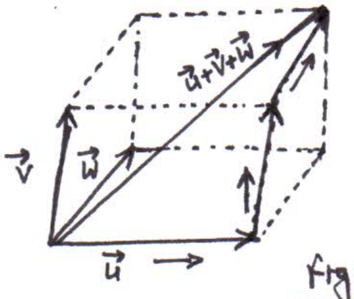
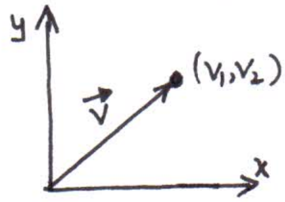
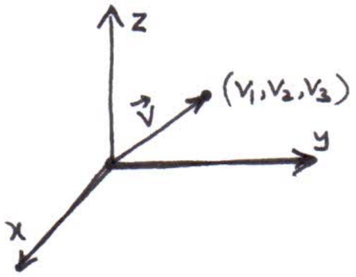


fig (iii)

VECTORS IN CO-ORDINATE SYSTEMS.

If a vector $\vec{v}$ in 2-space or 3-space is positioned with its initial point at the origin of a rectangular co-ordinate system, then the vector is completely determined by the co-ordinates of its terminal point (fig.). We call these co-ordinates the components of $\vec{v}$ relative to the co-ordinate system. We will write $\vec{v} = (v_1, v_2)$ to denote a vector $\vec{v}$ in 2-space with components (v_1, v_2), and $\vec{v} = (v_1, v_2, v_3)$ to denote a vector $\vec{v}$ in 3-space with components (v_1, v_2, v_3).





It should be evident geometrically that two vectors in 2-space or 3-space are Equivalent if and only if they have the same terminal point when their initial points are at the origin. Algebraically, this means that two vectors are Equivalent iff their corresponding components are equal. Thus, for example, the vectors

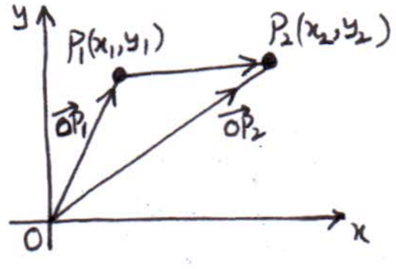
$\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ in 3-space are Equivalent iff $v_1 = w_1$, $v_2 = w_2$, $v_3 = w_3$.

VECTORS WHOSE INITIAL POINT IS NOT AT THE ORIGIN.

If $\overrightarrow{P_1P_2}$ denotes the vector with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then the components of this vector are given by

$$\overrightarrow{P_1P_2} = (x_2 - x_1,\ y_2 - y_1) \qquad \text{—— ①}$$

That is, the components of $\overrightarrow{P_1P_2}$ are obtained by subtracting the co-ordinates of the initial point from the co-ordinates of the terminal point.



for example, in fig. the vector $\overrightarrow{P_1P_2}$ is the difference of vectors $\overrightarrow{OP_2}$ and $\overrightarrow{OP_1}$, so

$$\begin{aligned}\overrightarrow{P_1P_2} &= \overrightarrow{OP_2} - \overrightarrow{OP_1} \\ &= (x_2, y_2) - (x_1, y_1) \\ &= (x_2 - x_1,\ y_2 - y_1)\end{aligned}$$

As you might expect, the components of a vector in 3-space that has initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$ are given by

$$\overrightarrow{P_1P_2} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1) \qquad \text{—— ②}$$

<u>Example</u> Find the components of a vector with initial point $P_1(2, -1, 4)$ & terminal point $P_2(7, 5, -8)$

<u>Soln.</u>
$$\begin{aligned}\overrightarrow{P_1P_2} &= (7-2,\ 5-(-1),\ -8-4) \\ &= (5, 6, -12).\end{aligned}$$

n-SPACE :- The set of real numbers can be viewed geometrically as a line. It is called the Real Line and is denoted by R or R^1. The superscript reinforces the intuitive idea that a line is one-dimensional. The set of all ordered pairs of real numbers (called 2-tuples) and the set of all ordered triples of real numbers (called 3-tuples) are denoted by R^2 and R^3, respectively. The superscript reinforces the idea that the ordered pairs correspond to points in the plane (two-dimensional) and ordered triples to points in space (three dimensional). The following defi. extends this idea.

Definition : If n is a positive integer, then an Ordered n-tuple is a sequence of n real numbers $(v_1, v_2, \dots, v_n)$. The set of all ordered n-tuples is called n-Space and denoted by R^n.

OPERATIONS ON VECTORS IN R^n

Our next objective is to define the operations of Addition, Subtraction and scalar multiplication for vectors in R^n. First we consider these operations on vectors in R^2 using components. If $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$, then

$$\vec{v} + \vec{w} = (v_1, v_2) + (w_1, w_2)$$

$$= (v_1 + w_1, v_2 + w_2) \quad \text{——— ①}$$

$$k\vec{v} = (kv_1, kv_2) \quad \text{——— ②}$$

In particular, it follows from ②

$$-\vec{v} = (-1)\vec{v}$$

$$= (-1)(v_1, v_2)$$

$$-\vec{v} = (-v_1, -v_2) \quad \text{——— ③}$$

and hence

$$\vec{w} - \vec{v} = \vec{w} + (-\vec{v})$$

$$= (w_1, w_2) + (-v_1, -v_2)$$

$$= (w_1 - v_1, w_2 - v_2) \quad \text{——— ④}$$

Motivated by formulas ①–④, we make the following definition —

Definition: If $\vec{v} = (v_1, v_2, \cdots, v_n)$ and $\vec{w} = (w_1, w_2, \dots, w_n)$ are vectors in R^n and if k is any scalar, then we define

$$\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n) \quad \text{——— ⑤}$$

$$k\vec{v} = (kv_1, kv_2, \dots, kv_n) \quad \text{——— ⑥}$$

$$-\vec{v} = (-v_1, -v_2, \dots, -v_n) \quad \text{——— ⑦}$$

$$\vec{w} - \vec{v} = \vec{w} + (-\vec{v}) = (w_1 - v_1, w_2 - v_2, \dots, w_n - v_n) \quad \text{——— ⑧}$$

In words, vectors are added (or subtracted) by adding (or subtracting) their corresponding components and a vector is multiplied by a scalar by multiplying each component by that scalar.

Example: If $\vec{v} = (1,-3,2)$ and $\vec{w} = (4,2,1)$, then find $\vec{v}+\vec{w}$, $2\vec{v}$, $-\vec{w}$ and $\vec{v}-\vec{w}$.

Soln. We have
$$\vec{v}+\vec{w} = (1,-3,2) + (4,2,1)$$
$$= (1+4, -3+2, 2+1) = (5,-1,3)$$
$$2\vec{v} = 2(1,-3,2) = (2,-6,4)$$
$$-\vec{w} = (-1)(4,2,1) = (-4,-2,-1)$$
and
$$\vec{v}-\vec{w} = \vec{v}+(-\vec{w})$$
$$= (1,-3,2) + (-4,-2,-1)$$
$$= (1-4, -3-2, 2-1) = (-3,-5,1)$$

THEOREM: If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in R^n, and if k and m are scalars, then:

(i) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$

(ii) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

(iii) $\vec{u}+\vec{0} = \vec{0}+\vec{u} = \vec{u}$

(iv) $\vec{u}+(-\vec{u}) = \vec{0}$

(v) $k(\vec{u}+\vec{v}) = k\vec{u}+k\vec{v}$

(vi) $(k+m)\vec{u} = k\vec{u}+m\vec{u}$

(vii) $k(m\vec{u}) = (km)\vec{u}$

(viii) $1.\vec{u} = \vec{u}$

LINEAR COMBINATIONS: If $\vec{w}$ is a vector in R^n, then $\vec{w}$ is said to be a Linear Combination of the vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r$ in R^n if it can be expressed in the form
$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r \qquad \text{———} \quad (1)$$
where $k_1, k_2, \cdots, k_r$ are scalars. These scalars are called the coefficients of the linear combination. In particular case, where $r=1$, formula (1) becomes
$$\vec{w} = k_1\vec{v}_1$$
i.e., A linear combination of a single vector is just a scalar multiple of that vector.

ALTERNATIVE NOTATIONS FOR VECTORS.

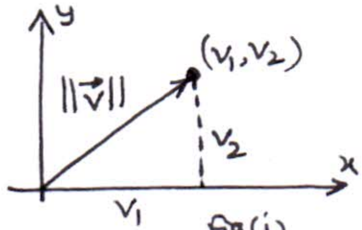
SEC 3.2 NORM, DOT PRODUCT and DISTANCE IN R^n

In this Section, we will be concerned with the notions of length and distance as they relate to vectors. We will first discuss these ideas in R^2 and R^3 and then extend them algebraically to R^n.

<u>NORM OF A VECTOR</u>: We will denote the length of a vector $\vec{v}$ by the symbol $\|\vec{v}\|$, which is read as the Norm of $\vec{v}$, the Length of $\vec{v}$ or the Magnitude of $\vec{v}$.

As suggested in Fig(i), it follows from the Theorem of Pythagoras that the Norm of a vector (v_1, v_2) in R^2 is

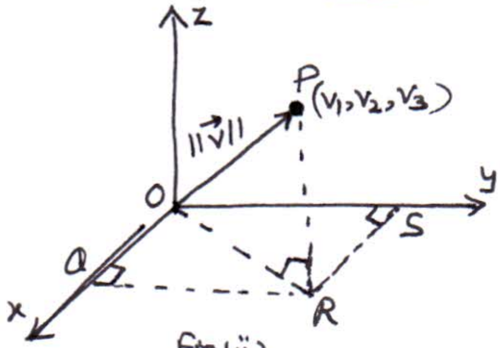
$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2} \qquad \text{——— ①}$$



Fig(i)

Similarly, for a vector (v_1, v_2, v_3) in R^3, it follows from Fig(ii) and two applications of the Theorem of Pythagoras that

$$\begin{aligned}
\|\vec{v}\|^2 &= (OP)^2 \\
&= (OR)^2 + (RP)^2 \\
&= (OQ)^2 + (QR)^2 + (RP)^2 \\
&= v_1^2 + v_2^2 + v_3^2
\end{aligned}$$



Fig(ii)

$$\therefore \|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \qquad \text{——— ②}$$

Motivated by the pattern of formulas ① & ②, we make the following definition —

<u>Definition</u>: If $\vec{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the Norm of $\vec{v}$ (also called the Length of $\vec{v}$ or the Magnitude of $\vec{v}$) is denoted by $\|\vec{v}\|$ and is defined by

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \qquad \text{——— ③}$$

<u>Example</u>: Find the norms of the vector $\vec{v} = (-3, 2, 1)$ in R^3 and the vector $W = (2, -1, 3, -5)$ in R^4.

<u>Soln.</u> We have
$$\begin{aligned}
\|\vec{v}\| &= \sqrt{(-3)^2 + (2)^2 + (1)^2} \\
&= \sqrt{9+4+1} = \sqrt{14}
\end{aligned}$$

and
$$\begin{aligned}
\|\vec{W}\| &= \sqrt{(2)^2 + (-1)^2 + (3)^2 + (-5)^2} \\
&= \sqrt{4+1+9+25} = \sqrt{39}
\end{aligned}$$

THEOREM: If $\vec{v}$ is a vector in R^n and if k is any scalar, then

(i) $\|\vec{v}\| \geq 0$

(ii) $\|\vec{v}\| = 0$ iff $\vec{v} = 0$

(iii) $\|k\vec{v}\| = |k| . \|\vec{v}\|$

UNIT VECTORS. A vector of norm '1' is called a Unit Vector. We can obtain a Unit Vector in a desired direction by choosing any non-zero vector $\vec{v}$ in that direction and multiplying $\vec{v}$ by the reciprocal of its length.

For example, if $\vec{v}$ is a vector of length 2 in R^2 or R^3, then $\frac{1}{2}\vec{v}$ is a Unit Vector in the same direction as $\vec{v}$.

More generally, if $\vec{v}$ is any non-zero vector in R^n, then $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$ —① defines a Unit Vector in the same direction as $\vec{v}$.

The process of multiplying a non-zero vector by the reciprocal of its length to obtain a unit vector is called normalizing $\vec{v}$.

Example: Find the Unit Vector $\vec{u}$ that has the same direction as $\vec{v} = (2, 2, -1)$.

Soln. The length of vector $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{(2)^2 + (2)^2 + (-1)^2}$$
$$= \sqrt{9} = 3$$

Thus, from ①

$$\vec{u} = \frac{1}{\|\vec{v}\|} . \vec{v}$$
$$= \frac{1}{3}(2, 2, -1)$$
$$\vec{u} = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$$

NOTE: We can confirm that

$$\|\vec{u}\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2}$$
$$= \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}}$$
$$= \sqrt{\frac{9}{9}} = 1.$$

THE STANDARD UNIT VECTORS: When a rectangular co-ordinate system is introduced in R^2 or R^3, the unit vectors in the positive directions of the co-ordinate axes are called the Standard Unit Vectors.

In R^2, these unit vectors are denoted by $\hat{i} = (1,0)$ and $\hat{j} = (0,1)$
and in R^3 by $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$, and $\hat{k} = (0,0,1)$

Every vector $\vec{v} = (v_1, v_2)$ in R^2 and $\vec{v} = (v_1, v_2, v_3)$ in R^3 can be expressed as a linear combination of standard unit vectors by writing

$$\vec{v} = (v_1, v_2) = v_1(1,0) + v_2(0,1) = v_1\hat{i} + v_2\hat{j} \quad \text{--- (1)}$$

$$\vec{v} = (v_1, v_2, v_3) = v_1(1,0,0) + v_2(0,1,0) + v_3(0,0,1) = v_1\hat{i} + v_2\hat{j} + v_3\hat{k} \quad \text{--- (2)}$$

Moreover, we can generalize these formulas to R^n by defining standard unit vectors in R^n to be

$$\hat{e}_1 = (1,0,0,\cdots,0),\ \hat{e}_2 = (0,1,0,\cdots,0),\ \cdots,\ e_n = (0,0,0,\cdots,1) \quad \text{--- (3)}$$

in which case, every vector $\vec{v} = (v_1, v_2, \cdots, v_n)$ in R^n can be expressed as

$$\vec{v} = (v_1, v_2, \cdots, v_n) = v_1\hat{e}_1 + v_2\hat{e}_2 + \cdots\cdots + v_n\hat{e}_n \quad \text{--- (4)}$$

Example (Linear Combinations of Standard Unit Vectors)

$$(2,-3,4) = 2\hat{i} - 3\hat{j} + 4\hat{k}, \text{ where } \hat{i} = (1,0,0),\ \hat{j} = (0,1,0)\ \&\ \hat{k} = (0,0,1)$$

& $(7,3,-4,5) = 7\hat{e}_1 + 3\hat{e}_2 - 4\hat{e}_3 + 5\hat{e}_4$, where $\hat{e}_1 = (1,0,0,0)$, $\hat{e}_2 = (0,1,0,0)$, etc.

DISTANCE IN R^n: If P_1 and P_2 are points in R^2 or R^3, then the length of vector $\overrightarrow{P_1P_2}$ is equal to the distance d between the points P_1 and P_2.

The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in 2-space is

$$d = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \quad \text{--- (1)}$$

and The distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in 3-space is

$$d = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} \quad \text{--- (2)}$$

Motivated by formulas (1) & (2), we make the following definition —

Definition: If $\vec{u} = (u_1, u_2, \cdots, u_n)$ and $\vec{v} = (v_1, v_2, \cdots, v_n)$ are points in R^n, then we denote the distance between $\vec{u}$ and $\vec{v}$ by $d(\vec{u}, \vec{v})$ and define it to be

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + \cdots + (u_n-v_n)^2} \quad \text{--- (3)}$$

Example: find the distance between $\vec{u} = (1,3,-2,7)$ and $\vec{v} = (0,7,2,2)$.

Soln: The distance between $\vec{u}$ and $\vec{v}$ is

$$d(\vec{u}, \vec{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2}$$

$$= \sqrt{1 + 16 + 16 + 25}$$

$$= \sqrt{58}$$

DOT PRODUCT : Our next objective is to define a useful multiplication operation on vectors in R^2 and R^3 and then extend that operation to R^n.

Definition : If $\vec{u}$ and $\vec{v}$ are non-zero vectors in R^2 or R^3, and if θ is the angle between $\vec{u}$ and $\vec{v}$, then the Dot Product (also called Euclidean Inner Product) of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u}.\vec{v}$ and is defined as

$$\vec{u}.\vec{v} = \|\vec{u}\|\,\|\vec{v}\| \cos\theta \quad , \text{where } 0 \le \theta \le \pi \qquad \text{———} \quad ①$$

If $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$, then we define $\vec{u}.\vec{v}$ to be 0.

The sign of the dot product reveals information about the angle θ that we can obtain by rewriting formula ① as

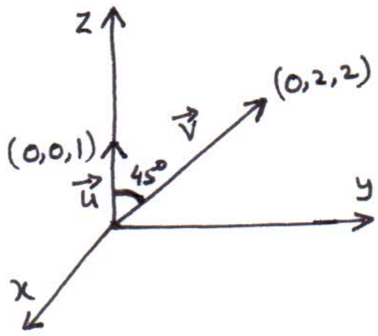
$$\cos\theta = \frac{\vec{u}.\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} \qquad \text{———} \quad ②$$

Since $0 \le \theta \le \pi$, it follows from ② that

i) If $\vec{u}.\vec{v} > 0$, then θ is acute

ii) If $\vec{u}.\vec{v} < 0$, then θ is obtuse

iii) If $\vec{u}.\vec{v} = 0$, then $\theta = \frac{\pi}{2}$.



Example ① Find the dot product of the vectors shown in fig.

Soln. Given that $\vec{u} = (0,0,1)$, $\vec{v} = (0,2,2)$ & $\theta = 45°$

The lengths of vectors $\vec{u}$ & $\vec{v}$ are

$$\|\vec{u}\| = \sqrt{0+0+(1)^2} = 1 \quad \text{and} \quad \|\vec{v}\| = \sqrt{0+(2)^2+(2)^2} = \sqrt{8} = 2\sqrt{2}$$

Thus, from formula ①,

$$\begin{aligned} \vec{u}.\vec{v} &= \|\vec{u}\|\,\|\vec{v}\| \cos\theta \\ &= (1)(2\sqrt{2}) \cos 45° \\ &= 2\sqrt{2} \times \frac{1}{\sqrt{2}} = 2 \end{aligned}$$

COMPONENT FORM OF THE DOT PRODUCT

Definition : If $\vec{u} = (u_1, u_2, \cdots, u_n)$ and $\vec{v} = (v_1, v_2, \cdots, v_n)$ are vectors in R^n, then Dot Product (also called Euclidean Inner Product) of $\vec{u}$ & $\vec{v}$ is denoted by $\vec{u}.\vec{v}$ and is defined by

$$\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n \qquad \text{—— (1)}$$

NOTE : If $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ are vectors in R^2,

then $\vec{u}.\vec{v} = u_1v_1 + u_2v_2$ —— (2)

and if $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in R^3,

then $\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$ —— (3)

Example (i) compute the dot product of the vectors $\vec{u} = (0,0,1)$ and $\vec{v} = (0,2,2)$

(ii) Calculate $\vec{u}.\vec{v}$ for the following vectors in R^4: $\vec{u} = (-1,3,5,7)$ and $\vec{v} = (-3,-4,1,0)$

Soln. (i) Here $\vec{u} = (0,0,1)$ and $\vec{v} = (0,2,2)$

$\therefore \quad \vec{u}.\vec{v} = (0)(0) + (0)(2) + (1)(2) = 2$

(ii) Here $\vec{u} = (-1,3,5,7)$ and $\vec{v} = (-3,-4,1,0)$

$\therefore \quad \vec{u}.\vec{v} = (-1)(-3) + (3)(-4) + (5)(1) + (7)(0)$

$= 3 - 12 + 5 + 0 = -4.$

ALGEBRAIC PROPERTIES OF THE DOT PRODUCT

In the special case where $\vec{u} = \vec{v}$ in above Definition, we obtain

$$\vec{v}.\vec{v} = v_1^2 + v_2^2 + \cdots + v_n^2$$
$$= \|\vec{v}\|^2$$

ie., $\|\vec{v}\| = \sqrt{\vec{v}.\vec{v}}$

This yields the formula for expressing the length of a vector in terms of dot product.

Dot Products have many of the same algebraic properties as products of real numbers

THEOREM. If $\vec{u}, \vec{v}$ & $\vec{w}$ are vectors in R^n, and if k is a scalar, then

(i) $\vec{u}.\vec{v} = \vec{v}.\vec{u}$ (Symmetry Property)

(ii) $\vec{u}.(\vec{v}+\vec{w}) = \vec{u}.\vec{v} + \vec{u}.\vec{w}$ (Distributive Property)

(iii) $k(\vec{u}.\vec{v}) = (k\vec{u}).\vec{v}$ (Homogeneity Property)

(iv) $\vec{v}.\vec{v} \geq 0$ and $\vec{v}.\vec{v} = 0$ iff $\vec{v} = \vec{0}$ (Positivity Property)

THEOREM: If $\vec{u}, \vec{v}$ and $\vec{w}$ are vectors in R^n, and if k is a scalar, then:

(i) $\vec{0} \cdot \vec{v} = \vec{v} \cdot \vec{0} = 0$

(ii) $(\vec{u}+\vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

(iii) $\vec{u} \cdot (\vec{v}-\vec{w}) = \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{w}$

(iv) $(\vec{u}-\vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} - \vec{v} \cdot \vec{w}$

(v) $k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$

Example (Calculating with Dot Products):- Calculate $(\vec{u}-2\vec{v}) \cdot (3\vec{u}+4\vec{v})$

Soln.
$$(\vec{u}-2\vec{v}) \cdot (3\vec{u}+4\vec{v}) = \vec{u} \cdot (3\vec{u}+4\vec{v}) - 2\vec{v} \cdot (3\vec{u}+4\vec{v})$$
$$= 3(\vec{u} \cdot \vec{u}) + 4(\vec{u} \cdot \vec{v}) - 6(\vec{v} \cdot \vec{u}) - 8(\vec{v} \cdot \vec{v})$$
$$= 3\|\vec{u}\|^2 - 2(\vec{u} \cdot \vec{v}) - 8\|\vec{v}\|^2$$

SEC 3.3 ORTHOGONALITY

In this Section, we will focus on the notion of 'perpendicularity'. Perpendicular vectors in R^n play an important role in a wide variety of applications.

ORTHOGONAL VECTORS: Recall from the previous Section that the angle θ between two non-zero vectors $\vec{u}$ and $\vec{v}$ in R^n is defined by the formula

$$\theta = \cos^{-1}\left(\frac{\vec{u}.\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)$$

It follows from this that $\theta = \frac{\pi}{2}$ iff $\vec{u}.\vec{v} = 0$. Thus, we make the following Definition

Definition: Two non-zero vectors $\vec{u}$ and $\vec{v}$ in R^n are said to be Orthogonal or perpendicular if $\vec{u}.\vec{v} = 0$. We also agree that zero vector in R^n is Orthogonal to every vector in R^n.

A non-empty set of vectors in R^n is called an Orthogonal set if all pairs of distinct vectors in the set are Orthogonal.

An orthogonal set of unit vectors is called an Orthonormal set.

Example (i) Show that $\vec{u} = (-2,3,1,4)$ and $\vec{v} = (1,2,0,-1)$ are orthogonal vectors in R^4.

(ii) Show that the set $S = \{\hat{i},\hat{j},\hat{k}\}$ of standard unit vectors is an orthogonal set in R^3.

Soln. (i) Here $\vec{u}.\vec{v} = (-2,3,1,4).(1,2,0,-1)$

$$= (-2)(1) + (3)(2) + (1)(0) + (4)(-1)$$

$$= -2 + 6 + 0 - 4 = 0$$

Hence the vectors $\vec{u}$ and $\vec{v}$ are Orthogonal.

(ii) The set $S = \{\hat{i},\hat{j},\hat{k}\}$ will be Orthogonal if all pairs of distinct vectors are Orthogonal, that is, $\hat{i}.\hat{j} = \hat{i}.\hat{k} = \hat{j}.\hat{k} = 0$.

Now $\hat{i}.\hat{j} = (1,0,0).(0,1,0)$

$$= (1)(0) + (0)(1) + (0)(0) = 0$$

$\hat{i}.\hat{k} = (1,0,0).(0,0,1)$

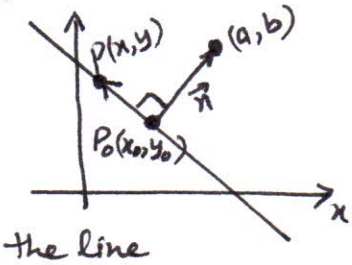
$$= (1)(0) + (0)(0) + (0)(1) = 0$$

& $\hat{j}.\hat{k} = (0,1,0).(0,0,1)$

$$= (0)(0) + (1)(0) + (0)(1) = 0$$

Hence, the set $S = \{\hat{i},\hat{j},\hat{k}\}$ is Orthogonal.

LINES AND PLANES DETERMINED BY POINTS AND NORMALS.



Fig(i) shows the line through the point $P_0(x_0, y_0)$ that has normal $\vec{n} = (a,b)$.

The line is represented by vector equ.

$\vec{n} \cdot \overrightarrow{P_0P} = 0$, where P is arbitrary point (x,y) on the line

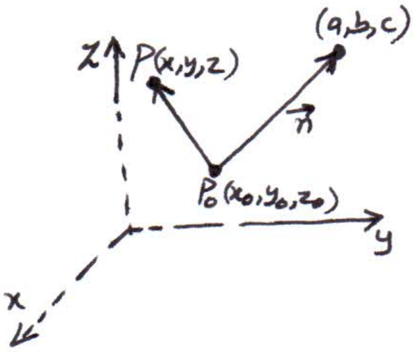
$\Rightarrow \quad (a,b) \cdot (x-x_0, y-y_0) = 0$

$\Rightarrow \quad a(x-x_0) + b(y-y_0) = 0 \quad \text{——} ①$

This is called the Point-Normal equ. of the line.

Similarly, the point-normal equ. of the plane is given as

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \quad \text{——} ②$



Example : It follows from ① that in R^2, the equ. $6(x-3) + (y+7) = 0$ represents the line through the point $(3,-7)$ with normal $\vec{n} = (6,1)$.
and it follows from ② that in R^3, the equ. $4(x-3) + 2y - 5(z-7) = 0$ represents the plane through the point $(3,0,7)$ with normal $\vec{n} = (4,2,-5)$.

When convenient, the terms in equations ① & ② can be multiplied out and the constants combined. This leads to the following theorem —

THEOREM. (i) If a & b are constants that are not both zero, then an equ. of the form $\quad ax + by + c = 0 \quad \text{——} ③$
represents a line in R^2 with normal $\vec{n} = (a,b)$.

(ii) If a, b & c are constants that are not all zero, then an equ. of the form $\quad ax + by + cz + d = 0 \quad \text{——} ④$
represents a plane in R^3 with normal $\vec{n} = (a,b,c)$.

ORTHOGONAL PROJECTIONS. In many applications, it is necessary to 'decompose' a vector $\vec{u}$ into a sum of two terms, one term being a scalar multiple of a specified non-zero vector $\vec{a}$ and the other term being orthogonal to $\vec{a}$.

For example, if $\vec{u}$ and $\vec{a}$ are vectors in R^2 that are positioned so that their initial points coincide at a point Q, then we can create such a decomposition as follows (See Fig.)

* Drop a perpendicular from the tip of $\vec{u}$ to the line through $\vec{a}$.
* Construct the vector $\vec{w}_1$ from Q to the foot of the perpendicular.
* Construct the vector $\vec{w}_2 = \vec{u} - \vec{w}_1$.

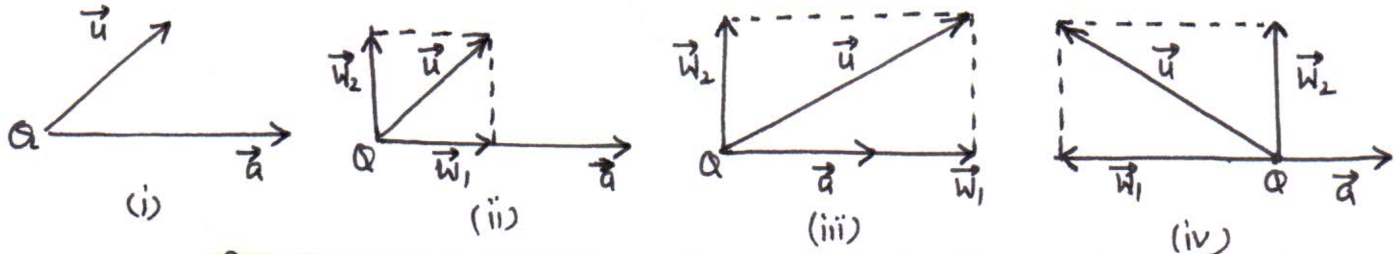


Fig. In parts (ii) through (iv), $\vec{u} = \vec{w}_1 + \vec{w}_2$, where $\vec{w}_1$ is parallel to $\vec{a}$ and $\vec{w}_2$ is orthogonal to $\vec{a}$

Since $\vec{w}_1 + \vec{w}_2 = \vec{w}_1 + (\vec{u} - w_1) = \vec{u}$

we have decomposed $\vec{u}$ into a sum of two orthogonal vectors, the first term being a scalar multiple of $\vec{a}$ and the second being orthogonal to $\vec{a}$.

The following theorem shows that the foregoing results, which we illstrated using vectors in R^2, apply as well in R^n.

THEOREM (PROJECTION THEOREM)

If $\vec{u}$ and $\vec{a}$ are vectors in R^n, and if $\vec{a} \neq \vec{0}$, then $\vec{u}$ can be expressed in exactly one way in the form $\vec{u} = \vec{w}_1 + \vec{w}_2$, where $\vec{w}_1$ is a scalar multiple of $\vec{a}$ and $\vec{w}_2$ is orthogonal to $\vec{a}$.

NOTE. The vector $\vec{w}_1$ is called the Orthogonal Projection of $\vec{u}$ on $\vec{a}$ or sometimes the Vector Component of $\vec{u}$ along $\vec{a}$ and is commonly denoted by the symbol $\mathrm{proj}_{\vec{a}}\vec{u}$; and the vector $\vec{w}_2$ is called the Vector Component of $\vec{u}$ Orthogonal to $\vec{a}$ and is commonly denoted by $\vec{u} - \mathrm{proj}_{\vec{a}}\vec{u}$.

By Projection Theorem, we have

$$\vec{w}_1 = \mathrm{proj}_{\vec{a}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\right)\vec{a} \qquad \text{[Vector Component of } \vec{u} \text{ along } \vec{a}]$$

$$\vec{w}_2 = \vec{u} - \mathrm{proj}_{\vec{a}}\vec{u} = \vec{u} - \left(\frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\right)\vec{a} \qquad \text{[Vector Component of } \vec{u} \text{ Orthogonal to } \vec{a}]$$

Example: Find the vector component of $\vec{u}$ along $\vec{a}$ and the vector component of $\vec{u}$ orthogonal to $\vec{a}$, where $\vec{u} = (2,-1,3)$ and $\vec{a} = (4,-1,2)$.

Soln. Here $\vec{u} = (2,-1,3)$ and $\vec{a} = (4,-1,2)$

$\therefore$
$$\vec{u}\cdot\vec{a} = (2)(4) + (-1)(-1) + (3)(2)$$
$$= 15$$
$$\|\vec{a}\|^2 = (4)^2 + (-1)^2 + (2)^2$$
$$= 21$$

Thus the vector component of $\vec{u}$ along $\vec{a}$ is

$$\text{proj}_{\vec{a}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\right)\vec{a}$$
$$= \frac{15}{21}(4,-1,2) = \frac{5}{7}(4,-1,2) = \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right)$$

and the vector component of $\vec{u}$ orthogonal to $\vec{a}$ is

$$\vec{u} - \text{proj}_{\vec{a}}\vec{u} = (2,-1,3) - \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right)$$
$$= \left(2-\frac{20}{7}, -1+\frac{5}{7}, 3-\frac{10}{7}\right) = \left(-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7}\right)$$

NOTE. As a check, we may verify that the vectors $\vec{u} - \text{proj}_{\vec{a}}\vec{u}$ and $\vec{a}$ are perpendicular by showing that their dot product is zero.

NOTE NORM OF VECTOR COMPONENT OF $\vec{u}$ along $\vec{a}$: —

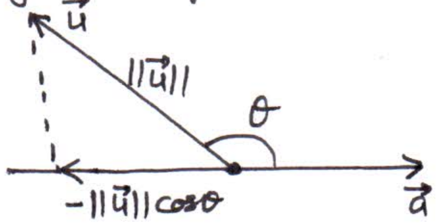
$$\|\text{proj}_{\vec{a}}\vec{u}\| = \left\|\left(\frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\right)\vec{a}\right\|$$
$$= \left|\frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\right|\|\vec{a}\|$$
$$= \frac{|\vec{u}\cdot\vec{a}|}{\|\vec{a}\|^2}\|\vec{a}\|$$
$$= \frac{|\vec{u}\cdot\vec{a}|}{\|\vec{a}\|} \quad \text{——— } ①$$

If θ denotes the angle between $\vec{u}$ & $\vec{a}$, then $\vec{u}\cdot\vec{a} = \|\vec{u}\|\,\|\vec{a}\|\cos\theta$

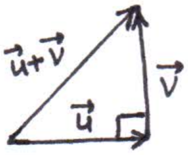
so ① can be written as $\quad \|\text{proj}_{\vec{a}}\vec{u}\| = \|\vec{u}\|\,|\cos\theta| \quad$ ——— ②

A geometric interpretation of this result is given in fig.

(i) $0 \le \theta < \frac{\pi}{2}$



(ii) $\frac{\pi}{2} < \theta \le \pi$.



THEOREM OF PYTHAGORAS IN R^n

If $\vec{u}$ and $\vec{v}$ are Orthogonal vectors in R^n with Euclidean inner product, then

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Example : Verify the Theorem of Pythagoras for the vectors $\vec{u} = (-2,3,1,4)$ & $\vec{v} = (1,2,0,-1)$.

Soln. Here $\vec{u}\cdot\vec{v} = (-2)(1) + (3)(2) + (1)(0) + (4)(-1) = 0$

$$\|\vec{u}\|^2 = (-2)^2 + (3)^2 + (1)^2 + (4)^2 = 30 \quad \text{——— ①}$$

$$\|\vec{v}\|^2 = (1)^2 + (2)^2 + (0)^2 + (-1)^2 = 6 \quad \text{——— ②}$$

$$\vec{u}+\vec{v} = (-2+1,\ 3+2,\ 1+0,\ 4-1)$$

$$= (-1, 5, 1, 3)$$

$$\therefore \|\vec{u}+\vec{v}\|^2 = (-1)^2 + (5)^2 + (1)^2 + (3)^2 = 36 \quad \text{——— ③}$$

from ①, ② & ③ we have $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

THEOREM (i) In R^2, the distance D between the point $P_0(x_0, y_0)$ and line $ax+by+c=0$ is given by

$$D = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$$

(ii) In R^3, the distance D between the point $P_0(x_0,y_0,z_0)$ and plane $ax+by+cz+d=0$ is given by

$$D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

Example : Find the distance D between the point $(1,-4,-3)$ and the plane $2x-3y+6z=-1$.

Soln. First rewrite the eqn. of the plane as

$$2x-3y+6z+1 = 0$$

from which we obtain

$$D = \frac{|2(1)+(-3)(-4)+6(-3)+1|}{\sqrt{(2)^2+(-3)^2+(6)^2}}$$

$$= \frac{|-3|}{7} = \frac{3}{7}$$

SEC (3.4) THE GEOMETRY OF LINEAR SYSTEMS.

In this Section, we will use parametric and vector methods to study general systems of linear equations.

VECTOR AND PARAMETRIC EQUATIONS OF LINES IN R^2 and R^3

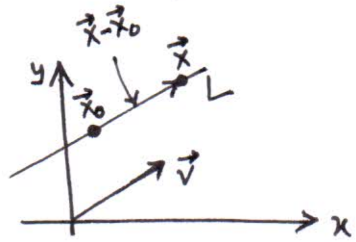
THEOREM : Let L be the line in R^2 or R^3 that consists the point $\vec{x}_0$ and is parallel to the non-zero vector $\vec{v}$. Then the equ. of the line through $\vec{x}_0$ that is parallel to $\vec{v}$ is

$$\vec{x} = \vec{x}_0 + t\vec{v} \qquad \text{—— (1)}$$

If $\vec{x}_0 = \vec{0}$, then the line passes through the origin and the equ. has the form

$$\vec{x} = t\vec{v} \qquad \text{—— (2)}$$

where the parameter t varies from $-\infty$ to ∞.



Proof - Let L is the line that contains the point $\vec{x}_0$ and is parallel to $\vec{v}$. If $\vec{x}$ is a general point on such a line, then as illustrated in fig(i), the vector $\vec{x}-\vec{x}_0$ will be some scalar multiple of $\vec{v}$, say

$$\vec{x}-\vec{x}_0 = t\vec{v}$$

or equivalently $\quad \vec{x} = \vec{x}_0 + t\vec{v}$

As the variable t (called parameter) varies from $-\infty$ to ∞, the point $\vec{x}$ traces out the line L.

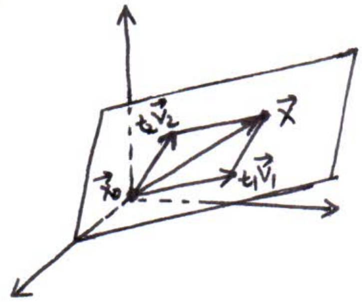
VECTOR AND PARAMETRIC EQUATIONS OF PLANES IN R^3

THEOREM : Let W be the plane in R^3 that contains the point $\vec{x}_0$ and is parallel to the non-collinear vectors $\vec{v}_1$ & $\vec{v}_2$. Then an equ. of the plane through $\vec{x}_0$ and parallel to $\vec{v}_1$ & $\vec{v}_2$ is

$$\vec{x} = \vec{x}_0 + t_1\vec{v}_1 + t_2\vec{v}_2 \qquad \text{—— (3)}$$

If $\vec{x}_0 = \vec{0}$, then the plane passes through the origin and the equ. has the form

$$\vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2 \qquad \text{—— (4)}$$



Proof Let W is the plane that contains the point $\vec{x}_0$ and is parallel to the non-collinear vectors $\vec{v}_1$ & $\vec{v}_2$.
As shown in fig.(ii), if $\vec{x}$ is any point in the plane, then by forming suitable scalar multiples of $\vec{v}_1$ & $\vec{v}_2$, say $t_1\vec{v}_1$ & $t_2\vec{v}_2$, we can create a parallelogram with diagonal $\vec{x}-\vec{x}_0$ and adjacent sides $t_1\vec{v}_1$ & $t_2\vec{v}_2$. Thus, we have

$$\vec{x}-\vec{x}_0 = t_1\vec{v}_1 + t_2\vec{v}_2$$

or equivalently, $\quad \vec{x} = \vec{x}_0 + t_1\vec{v}_1 + t_2\vec{v}_2$

As the variables t_1 and t_2 (called parameters) vary independently from $-\infty$ to ∞, the point $\vec{x}$ varies over the entire plane W.

Motivated by the forms of formulas ① to ④, we can extend the notions of line and plane to R^n by making the following definitions —

<u>Definition ①</u> If $\vec{x}_0$ and $\vec{v}$ are vectors in R^n and if $\vec{v}$ is non-zero, then the equ.

$$\vec{x} = \vec{x}_0 + t\vec{v} \quad \text{—— ⑤}$$

defines the line through $\vec{x}_0$ that is parallel to $\vec{v}$.

In the special case where $\vec{x}_0 = \vec{0}$, the line is said to pass through the origin.

<u>Definition ②</u> If $\vec{x}_0, \vec{v}_1$ and $\vec{v}_2$ are vectors in R^n and if $\vec{v}_1$ & $\vec{v}_2$ are not collinear, then the equ. $\vec{x} = \vec{x}_0 + t_1\vec{v}_1 + t_2\vec{v}_2$ —— ⑥

defines the plane through $\vec{x}_0$ that is parallel to $\vec{v}_1$ & $\vec{v}_2$.

In the special case where $\vec{x}_0 = \vec{0}$, the plane is said to pass through the origin.

<u>NOTE</u> - Equations ⑤ & ⑥ are called Vector forms of a line and plane in R^n. If the vectors in these equations are expressed in terms of their components and the corresponding components on each side are equated, then the resulting equations are called Parametric Equations of the line and plane.

<u>Example ① (Vector and Parametric Equations of Lines in R^2 and R^3)</u>

(i) Find a vector equation and parametric equations of the line in R^2 that passes through the origin and is parallel to the vector $\vec{v} = (-2,3)$.

(ii) Find a vector equ. and parametric equations of the line in R^3 that passes through the point $P_0(1,2,-3)$ and is parallel to the vector $\vec{v} = (4,-5,1)$.

(iii) Use the vector equ. obtained in part(ii) to find two points on line that are different from P_0.

<u>Solution</u> (i) The vector equ. of the line passing through origin is $\vec{x} = t\vec{v}$ —— ①

If we let $\vec{x} = (x,y)$ in R^2 then equ. ① can be expressed in vector form as

$$(x,y) = t(-2,3) \quad \text{—— ②} \quad (\text{Vector Equ. of the line})$$

$$\Rightarrow \quad (x,y) = (-2t, 3t)$$

Equating corresp. components, $x = -2t$, $y = 3t$ (parametric equ. of the required line)

(ii) The vector equ. of the line passing through the point x_0 and parallel to vector $\vec{v}$ is given by $\vec{x} = \vec{x}_0 + t\vec{v}$ —— ①

If we let $\vec{x} = (x,y,z)$ in R^3 and if we take $\vec{x}_0 = (1,2,-3)$, then equ. ① can be expressed in vector form as

$$(x,y,z) = (1,2,-3) + t(4,-5,1) \quad \text{—— ② (Vector Equ.)}$$

$$= (1+4t,\ 2-5t,\ -3+t) \quad \text{—— ③}$$

Equating corresp. components, $x = 1+4t$, $y = 2-5t$, $z = -3+t$ (parametric equ. of required line)

(iii) A point on the line represented by equ. ③ can be obtained by substituting a specific numerical value for 't'. However, since $t=0$ produces $(x,y,z) = (1,2,-3)$ which is the point P_0, this value of t does not serve our purpose. Taking $t=1$ produces the point $(5,-3,-2)$ and taking $t=-1$ produces the point $(-3,7,-4)$. Any other distinct values for t (except $t=0$) would work just as well.

Example ② (Vector and Parametric Equ. of a Plane in R^3)

Find vector and parametric equations of the plane $x-y+2z=5$.

Soln. We will find the parametric equations first. We can do this by solving the equ. for any one of the variables in terms of the other two and then using those two variables as parameters. For example, solving the given equ. for x in terms of y & z yields

$$x = 5+y-2z \quad \text{———} \quad ①$$

and then using y & z as parameters t_1 & t_2, respectively, yields

$$x = 5+t_1-2t_2$$

Thus, parametric equ. are $\quad x=5+t_1-2t_2\ ,\ y=t_1\ ,\ z=t_2$

To obtain a vector equ. of the plane, we rewrite these parametric equations as

$$\begin{aligned}(x,y,z) &= (5+t_1-2t_2,\ t_1,\ t_2)\\ &= (5+t_1-2t_2\ ,\ 0+1t_1+0t_2\ ,\ 0+0t_1+1t_2)\\ &= (5,0,0)+t_1(1,1,0)+t_2(-2,0,1) \quad \text{———} \quad ②\end{aligned}$$

which is of the vector form $\vec{x}=\vec{x}_0+t_1\vec{v}_1+t_2\vec{v}_2$

Example ③ (Vector and Parametric Equ. of Lines and Planes in R^4)

(i) Find Vector and parametric equations of the line through the origin of R^4 that is parallel to the vector $\vec{v}=(5,-3,6,1)$.

(ii) Find vector and parametric equations of the plane in R^4 that passes through the point $\vec{x}_0=(2,-1,0,3)$ and is parallel to both $\vec{v}_1=(1,5,2,-4)$ and $\vec{v}_2=(0,7,-8,6)$.

Soln. (i) If we let $\vec{x}=(x_1,x_2,x_3,x_4)$, then the vector equ. of the line passing through origin and parallel to $\vec{v}$ is given by $\quad \vec{x}=t\vec{v}$

$$\text{i.e.,} \quad (x_1,x_2,x_3,x_4) = t(5,-3,6,1)$$

Equating corresponding components yields the parametric equations as

$$x_1=5t\ ,\ x_2=-3t\ ,\ x_3=6t\ ,\ x_4=t.$$

(ii) The vector equ. of the plane passing through the point $\vec{x}_0$ and parallel to $\vec{v}_1$ & $\vec{v}_2$ is

$$\vec{x}=\vec{x}_0+t_1\vec{v}_1+t_2\vec{v}_2 \quad \text{———} \quad ①$$

If we let $\vec{x}=(x_1,x_2,x_3,x_4)$ in R^4 and if we take $\vec{x}_0=(2,-1,0,3)$, then equ. ① can be expressed as

$$(x_1,x_2,x_3,x_4) = (2,-1,0,3)+t_1(1,5,2,-4)+t_2(0,7,-8,6)$$

$$\Rightarrow \quad (x_1,x_2,x_3,x_4) = (2,-1,0,3)+(t_1,5t_1,2t_1,-4t_1)+(0,7t_2,-8t_2,6t_2)$$

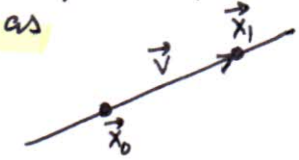
$$\Rightarrow \quad (x_1,x_2,x_3,x_4) = (2+t_1,-1+5t_1+7t_2,\ 2t_1-8t_2,\ 3-4t_1+6t_2)$$

Equating corresp. components yields the parametric equations as

$$\begin{aligned}x_1 &= 2+t_1\\ x_2 &= -1+5t_1+7t_2\\ x_3 &= 2t_1-8t_2\\ x_4 &= 3-4t_1+6t_2\end{aligned}$$

LINES THROUGH TWO POINTS IN R^n

If $\vec{x}_0$ and $\vec{x}_1$ are distinct point in R^n, then the line determined by these points is parallel to the vector $\vec{v} = \vec{x}_1 - \vec{x}_0$ so the line can be expressed in vector form as



$$\vec{x} = \vec{x}_0 + t(\vec{x}_1 - \vec{x}_0) \quad \text{—— (1)}$$

or equivalently as $\vec{x} = (1-t)\vec{x}_0 + t\vec{x}_1$ —— (2)

These are called two-point vector equations of a line in R^n.

Example (A line through two points in R^2)

Find vector and parametric equ. for the line in R^2 that passes through points $P(0,7)$ and $Q(5,0)$.

Solution: The vector equ. of a line passing through two points $\vec{x}_0$ & $\vec{x}_1$ is given as

$$\vec{x} = \vec{x}_0 + t(\vec{x}_1 - \vec{x}_0) \quad \text{—— (1)}$$

If we let $\vec{x} = (x,y)$ in R^2 and if we take $\vec{x}_0 = (0,7)$ & $\vec{x}_1 = (5,0)$, it follows that $\vec{x}_1 - \vec{x}_0 = (5,-7)$ and hence

$$(x,y) = (0,7) + t(5,-7) \quad \text{—— (2)} \quad \text{(Vector Equ. of line)}$$

$$= (0+5t,\ 7-7t)$$

Equating corresp. components yields the parametric equations as

$$x = 5t, \quad y = 7-7t \qquad \text{(Parametric Equ. of line)}$$

LINE SEGMENT JOINING THE POINTS.

If $\vec{x}_0$ and $\vec{x}_1$ are vectors in R^n, then the equ.

$$\vec{x} = \vec{x}_0 + t(\vec{x}_1 - \vec{x}_0), \quad 0 \le t \le 1 \quad \text{—— (1)}$$

defines the line segment from $\vec{x}_0$ to $\vec{x}_1$.

When convenient, Equ. (1) can be written as

$$\vec{x} = (1-t)\vec{x}_0 + t\vec{x}_1, \quad 0 \le t \le 1 \quad \text{—— (2)}$$

Example Find the equ. of line segment in R^2 from $\vec{x}_0 = (1,-3)$ to $\vec{x}_1 = (5,6)$.

Soln. The equ. of line segment is given by

$$\vec{x} = \vec{x}_0 + t(\vec{x}_1 - \vec{x}_0), \quad 0 \le t \le 1$$

Here $\vec{x}_0 = (1,-3)$, $\vec{x}_1 = (5,6)$ so that $\vec{x}_1 - \vec{x}_0 = (5-1, 6+3) = (4,9)$

Hence the equ. of line segment is

$$\vec{x} = (1,-3) + t(4,9).$$

SEC 3.5 CROSS PRODUCT

CROSS PRODUCT OF VECTORS – In Sec 3.2, we defined the dot product of two vectors $\vec{u}$ and $\vec{v}$ in n-space. That operation produced a Scalar as its result. We will now define a type of vector multiplication that produces a Vector as the result but which is applicable only to vectors in 3-space.

Definition 1 If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the Cross Product $\vec{u} \times \vec{v}$ is the vector defined by

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2\, ,\; u_3v_1 - u_1v_3\, ,\; u_1v_2 - u_2v_1)$$

or, in determinant notation,

$$\vec{u} \times \vec{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \qquad \text{—— } (1)$$

NOTE Instead of memorizing (1), we can obtain the components of $\vec{u} \times \vec{v}$ as follows –

* Form the 2×3 matrix $\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$, whose first row contains the components of $\vec{u}$ and second row contains the components of $\vec{v}$.
* To find the first component of $\vec{u} \times \vec{v}$, delete the first column and take the determinant; to find the second component, delete the second column and take negative of the determinant; and to find the third component, delete the third column and take the determinant.

Example 1 Find $\vec{u} \times \vec{v}$, where $\vec{u} = (1, 2, -2)$ and $\vec{v} = (3, 0, 1)$.

Solution: We have

$$\begin{bmatrix} 1 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \right)$$

$$= \big((1)(2) - 0\, ,\; -(1+6)\, ,\; (0-6) \big)$$

$$= (2, -7, -6)$$

THEOREM (Relationship Involving Cross Product and Dot Product)

If $\vec{u}, \vec{v}$ and $\vec{w}$ are vectors in 3-space, then

(i) $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ ($\vec{u} \times \vec{v}$ is Orthogonal to $\vec{u}$)

(ii) $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$ ($\vec{u} \times \vec{v}$ is Orthogonal to $\vec{v}$)

(iii) $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$ (Lagrange's Identity)

(iv) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$ (Relationship between Cross and Dot Product)

(v) $(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{v} \cdot \vec{w})\vec{u}$

Example ② ($\vec{u}\times\vec{v}$ is Perpendicular to $\vec{u}$ and to $\vec{v}$)

Consider the vectors $\vec{u}=(1,2,-2)$ and $\vec{v}=(3,0,1)$

Now $\vec{u}\times\vec{v}=(2,-7,-6)$, as shown in Example ①

Since $\vec{u}\cdot(\vec{u}\times\vec{v}) = (1)(2)+(2)(-7)+(-2)(-6)$

$= 2-14+12 = 0$

and $\vec{v}\cdot(\vec{u}\times\vec{v}) = (3)(2)+(0)(-7)+(1)(-6)$

$= 6+0-6 = 0$

$\therefore$ $\vec{u}\times\vec{v}$ is Orthogonal to both $\vec{u}$ and $\vec{v}$.

THEOREM : (Properties of Cross Product)

If $\vec{u}, \vec{v}$ and $\vec{w}$ are any vectors in 3-space and k is any scalar, then

(i) $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$

(ii) $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v}+\vec{u}\times\vec{w}$

(iii) $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w}+\vec{v}\times\vec{w}$

(iv) $k(\vec{u}\times\vec{v}) = (k\vec{u})\times\vec{v} = \vec{u}\times(k\vec{v})$

(v) $\vec{u}\times\vec{0} = \vec{0}\times\vec{u} = \vec{0}$

(vi) $\vec{u}\times\vec{u} = \vec{0}$

Example ③ (Standard Unit Vectors)

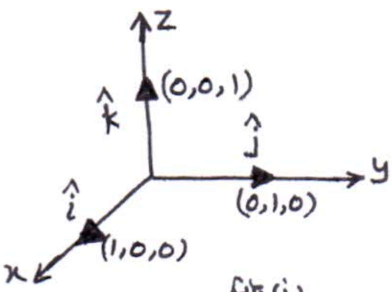


Fig (i)

Consider the vectors $\hat{i}=(1,0,0)$, $\hat{j}=(0,1,0)$, $\hat{k}=(0,0,1)$

These vectors each have length 1 and lie along the coordinate axes (See Fig(i)). They are called Standard Unit Vectors in 3-space.

Every vector $\vec{v}=(v_1,v_2,v_3)$ in 3-space is expressible in terms of $\hat{i}, \hat{j}$ & $\hat{k}$ since we can write

$$\vec{v}=(v_1,v_2,v_3) = v_1(1,0,0)+v_2(0,1,0)+v_3(0,0,1)$$
$$= v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$$

We obtain $\hat{i}\times\hat{j} = \left(\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right)$ $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right.$

$= (0,0,1) = \hat{k}$

We should have no trouble obtaining the following results —

$\hat{i}\times\hat{i}=0$	$\hat{j}\times\hat{j}=0$	$\hat{k}\times\hat{k}=0$
$\hat{i}\times\hat{j}=\hat{k}$	$\hat{j}\times\hat{k}=\hat{i}$	$\hat{k}\times\hat{i}=\hat{j}$
$\hat{j}\times\hat{i}=-\hat{k}$	$\hat{k}\times\hat{j}=-\hat{i}$	$\hat{i}\times\hat{k}=-\hat{j}$

Fig (ii)

Fig(ii) is helpful for remembering these results. Referring to this diagram, the cross product of two consecutive vectors going clockwise is the next vector around and the cross product of two consecutive vectors going counterclockwise is the negative of the next vector around.

DETERMINANT FORM OF CROSS PRODUCT

It is also worthnoting that a cross product can be represented symbolically in the form

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad , \text{ where } \vec{u}=(u_1,u_2,u_3) \ \& \ \vec{v}=(v_1,v_2,v_3)$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\hat{k} \qquad \text{——— (1)}$$

For example, if $\vec{u}=(1,2,-2)$ and $\vec{v}=(3,0,1)$ then

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}\hat{i} - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}\hat{j} + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}\hat{k}$$

$$= (2-0)\hat{i} - (1+6)\hat{j} + (0-6)\hat{k}$$

$$= 2\hat{i} - 7\hat{j} - 6\hat{k} = (2,-7,-6)$$

which agrees with the result obtained in Example (1)

NOTE : It is not true, in general, that $\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\times\vec{w}$.

For example, $\hat{i}\times(\hat{j}\times\hat{j}) = \hat{i}\times\vec{0} = \vec{0}$

and $(\hat{i}\times\hat{j})\times\hat{j} = \hat{k}\times\hat{j} = -\hat{i}$

Thus $\hat{i}\times(\hat{j}\times\hat{j}) \neq (\hat{i}\times\hat{j})\times\hat{j}$

NOTE : We know that $\vec{u}\times\vec{v}$ is Orthogonal to both $\vec{u}$ and $\vec{v}$. If $\vec{u}$ and $\vec{v}$ are non-zero vectors, it can be shown that the direction of $\vec{u}\times\vec{v}$ can be determined using the following "right hand rule" —

If θ be the angle between $\vec{u}$ and $\vec{v}$, and suppose $\vec{u}$ is rotated through the angle θ until it coincides with $\vec{v}$. If the fingures of right hand are cupped so that they point in the direction of rotation, then the thumb indicates (roughly) the direction of $\vec{u}\times\vec{v}$.

You may find it instructive to practice this rule with the products

$$\hat{i}\times\hat{j} = \hat{k} \ , \quad \hat{j}\times\hat{k} = \hat{i} \ , \quad \hat{k}\times\hat{i} = \hat{j}$$

GEOMETRIC INTERPRETATION OF CROSS PRODUCT

If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then the Norm of $\vec{u}\times\vec{v}$ has a useful geometric interpretation. Lagrange's identity states that

$$\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}.\vec{v})^2 \quad \text{——— ①}$$

If θ denotes the angle between $\vec{u}$ and $\vec{v}$

then $\quad \vec{u}.\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta \quad$ ——— ②

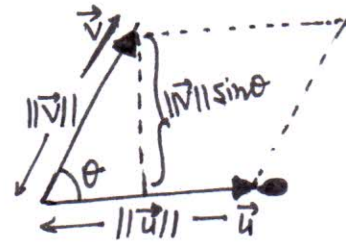
so ① can be rewritten as $\quad \|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - \|\vec{u}\|^2\|\vec{v}\|^2\cos^2\theta$

$$= \|\vec{u}\|^2\|\vec{v}\|^2(1-\cos^2\theta)$$

$$\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2\sin^2\theta$$

Since $0 \le \theta \le \pi$, it follows that $\sin\theta \geq 0$ so this can be rewritten as

$$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta \quad \text{——— ③}$$



But $\|\vec{v}\|\sin\theta$ is the altitude of parallelogram determined by $\vec{u}$ and $\vec{v}$ (See fig.). Thus

the Area A of this parallelogram is given by

$$A = (\text{base})(\text{altitude})$$

$$= \|\vec{u}\|\,\|\vec{v}\|\sin\theta$$

$$A = \|\vec{u}\times\vec{v}\| \rightarrow ④, \text{ using } ③$$

This result is even correct if $\vec{u}$ and $\vec{v}$ are collinear, since the parallelogram determined by $\vec{u}$ and $\vec{v}$ has zero Area and from ③ we have $\|\vec{u}\times\vec{v}\| = 0$, because $\theta = 0$ in this case.

THEOREM (Area of a Parallogram) — If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\|\vec{u}\times\vec{v}\|$ is equal to the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

Example ④ (Area of a Triangle)

Find the Area of a triangle determined by the points $P_1(2,2,0)$, $P_2(-1,0,2)$ and $P_3(0,4,3)$.

Solu. The area of triangle is $\frac{1}{2}$ the area of parallelogram determined by $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$.

Now $\quad \overrightarrow{P_1P_2} = (-1-2,\ 0-2,\ 2-0) = (-3,-2,2)$

$\quad\quad \overrightarrow{P_1P_3} = (0-2,\ 4-2,\ 3-0) = (-2,2,3)$

$$\therefore\ \overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3} = \left(\begin{vmatrix} -2 & 2 \\ 2 & 3 \end{vmatrix},\ -\begin{vmatrix} -3 & 2 \\ -2 & 3 \end{vmatrix},\ \begin{vmatrix} -3 & -2 \\ -2 & 2 \end{vmatrix}\right)$$

$$= \left((-6-4),\ -(-9+4),\ (-6-4)\right)$$

$$= (-10, 5, -10)$$

$$\therefore \|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}\| = \sqrt{(-10)^2+(5)^2+(-10)^2} = \sqrt{225} = 15$$

$\therefore$ Area of triangle, $A = \frac{1}{2}(\text{Area of parallelogram}) = \frac{1}{2}\|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}\| = \frac{15}{2}$

SCALAR TRIPLE PRODUCT — If $\vec{u}, \vec{v}$ & $\vec{w}$ are vectors in 3-space, then $\vec{u}\cdot(\vec{v}\times\vec{w})$ is called the Scalar Triple Product of $\vec{u}, \vec{v}$ & $\vec{w}$.

The Scalar Triple Product of $\vec{u}=(u_1,u_2,u_3)$, $\vec{v}=(v_1,v_2,v_3)$ and $\vec{w}=(w_1,w_2,w_3)$ can be calculated from the formula

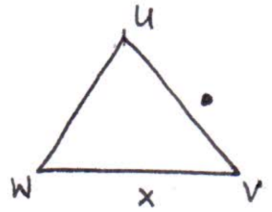
$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \quad \text{—— ①}$$

NOTE: The symbol $(\vec{u}\cdot\vec{v})\times\vec{w}$ makes no sense because we cannot form the cross product of a scalar (ie; $\vec{u}\cdot\vec{v}$) and a vector. Thus, no ambiguity arises if we write $\vec{u}\cdot\vec{v}\times\vec{w}$ rather than $\vec{u}\cdot(\vec{v}\times\vec{w})$. However, for clarity, we will usually keep the parentheses.

It follows from ① that

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \vec{w}\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\vec{w}\times\vec{u})$$

These relationships can be remembered by moving the vectors $\vec{u}, \vec{v}$ and $\vec{w}$ clockwise around the vertices of the triangle in Fig.



GEOMETRIC INTERPRETATION OF DETERMINANTS

The next theorem provides a useful geometric interpretation of 2x2 and 3x3 determinants.

THEOREM (i) The absolute value of the determinant $\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$ is equal to the Area of parallelogram in 2-space determined by the vectors $\vec{u}=(u_1,u_2)$ and $\vec{v}=(v_1,v_2)$.

(ii) The absolute value of the determinant $\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$

is equal to the volume of the parallelopiped in 3-space determined by the vectors $\vec{u}=(u_1,u_2,u_3)$, $\vec{v}=(v_1,v_2,v_3)$ and $\vec{w}=(w_1,w_2,w_3)$.

THEOREM: If the vectors $\vec{u}=(u_1,u_2,u_3)$, $\vec{v}=(v_1,v_2,v_3)$ and $\vec{w}=(w_1,w_2,w_3)$ have same initial point, then they lie in the same plane if and only if $\vec{u}\cdot(\vec{v}\times\vec{w})=0$ ie;, $\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$.