Chapter-8

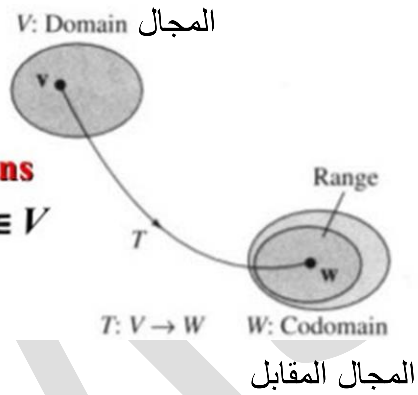
Linear Transformations

- A linear transformation is a function T that maps a vector space V into another vector space W:

$$T : V \xrightarrow{\text{mapping}} W, \quad V, W : \text{vector space}$$

V: the domain of T

W: the co-domain of T



متى نقول أنها تحويل خطي إذا تحقق الشرطين

Two axioms of linear transformations

(1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V$

(2) $T(c\mathbf{u}) = cT(\mathbf{u}), \quad \forall c \in R$

- Image of $\mathbf{v}$ under T:

If $\mathbf{v}$ is in V and $\mathbf{w}$ is in W such that

$$T(\mathbf{v}) = \mathbf{w}$$

Then $\mathbf{w}$ is called the image of $\mathbf{v}$ under T.

- **the range of T:**

The set of all images of vectors in V.

$$range(T) = \{T(\mathbf{v}) \mid \forall \mathbf{v} \in V\}$$

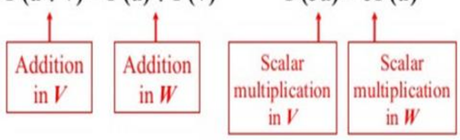
- the pre-image of $\mathbf{w}$:

The set of all $\mathbf{v}$ in V such that $T(\mathbf{v})=\mathbf{w}$.

- Notes:

(1) A **linear transformation** is said to be **operation preserving**.

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \qquad T(c\mathbf{u}) = cT(\mathbf{u})$$



(2) A linear transformation $T : V \to V$ from **a vector space into itself** is called a **linear operator**.

Q1. $T: R^2 \rightarrow R^2$ $v = (v_1, v_2) \in R^2$, $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$

a. Find the image of $v = (-1,2)$?
b. Find the pre-image of $w = (-1,11)$?

<u>Solution:</u>

a. $v = (-1,2)$

$$\mathrm{T(v)} = \mathrm{T}(-1,2) = (-1 - 2, -1 + 2(2))$$

$$= (-3,3)$$

b. $T(v) = w = (-1,11)$

We know $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1,11)$

نحول المسألة إلى معادلتين ... لنتمكن من إيجاد القيم

$$v_1 - v_2 = -1$$

$$-v_1 - 2v_2 = -11$$

Multiple -1 and Add

$$-3v_2 = -12$$

$$v_2 = \frac{-12}{-3} = 4$$

بالتعويض في المعادلة الأولي :

$$v_1 - 4 = -1$$

$$v_1 = -1 + 4 \Longrightarrow v_1 = 3$$

So (3,4) per-image of (-1,11)

Q1. Verify a linear Transformation T from R^2 *into* R^2

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

<u>Solution</u>:

Let $u = (u_1, u_2)$, $v = (v_1, v_2)$ *and* c *is any real number*

1. Let $u + v = (u_1, u_2) + (v_1, v_2)$
$= (u_1 + v_1, u_2 + v_2)$

$$T(u+v) = T(u_1+v_1\,, u_2+v_2)$$
$$= ((u_1+v_1) - (u_2+v_2)\,, (u_1+v_1) + 2(u_2+v_2))$$
$$= ((u_1-u_2) + (v_1-v_2)\,, (u_1+2u_2) + (v_1+2v_2))$$
$$= \left((u_1-u_2, u_1+2u_2) + (v_1-v_2, v_1+2v_2)\right)$$
$$= T(u) + T(v)$$

2. $cu = c(u_1, u_2) = (cu_1, cu_2)$

$$T(cu) = T(cu_1, cu_2) = (cu_1 - cu_2, cu_1 + 2cu_2)$$
$$= c(u_1 - u_2, u_1 + 2u_2)$$
$$= cT(u)$$

:. T is a linear transformation.

- Zero transformation :

$$T: V \to W \qquad T(v) = 0, \quad \forall v \in V$$

- Identity transformation:

$$T: V \to V \qquad T(v) = v, \quad \forall v \in V$$

Q1. Let $T: R^3 \to R^3$ *be a linear transformation* such that

$T(1,0,0) = (2,-1,4)$
$T(0,1,0) = (1,5,-2)$
$T(0,0,1) = (0,3,1)$

Find $T(2,3,-2)$?

Solution:

$$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$$

$$T(2,3,-2) = 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1)$$

$$= 2(2,-1,4) + 3(1,5,-2) - 2(0,3,-2)$$

$$= (7,7,0)$$

Q2. The function$T: R^3 \rightarrow R^3$ *is defined by*

$$T(u) = Av = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Find $T(u)$ *where* $v = (2, -1)$?

<u>Solution</u>:

$$v = (2, -1)$$

$$T(u) = Av = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

$$\therefore T(2, -1) = (6,3,0)$$