

# Week 7, chapter 4 (4.7 - 4.11)

## Row Space, Column Space and Null space

Row and Column Vectors of a  $2 \times 3$  Matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

The Row Vectors of  $A$  are

$$r_1 = [2 \ 1 \ 0] \text{ }^3 \text{ vectors} \text{ and } r_2 = [3 \ -1 \ 4] \text{ }^3 \text{ vectors}$$

and the column vectors of  $A$  are

$$c_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ }^2 \text{ vectors} \quad c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and } c_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Now we are going to define 3 important vector spaces that are associated with a matrix

definition: If  $A$  is an  $m \times n$  matrix, then the subspace of  $\mathbb{R}^n$  spanned by the row vectors of  $A$  is called the row space of  $A$ , and the subspace of  $\mathbb{R}^m$  spanned by the column vectors of  $A$  is called the column space of  $A$ .

The solution space of the homogeneous system of eqns  $AX=0$  which is a subspace of  $\mathbb{R}^n$ , is called the null space of  $A$ .

Q1. Find the basis for Row and Column spaces of the matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In the first row the

Soln: Since the matrix already is in Row echelon form. Therefore the non-zero vectors are

$$r_1 = (1 \ -2 \ 5 \ 0 \ 3)$$

$$r_2 = (0 \ 1 \ 3 \ 0 \ 0)$$

$$r_3 = (0 \ 0 \ 0 \ 1 \ 0)$$

Now find the basis for the column space.

In each row the first Non-zero element is called the leading element and it is 1, and corresponding to this leading element the variable is called leading variable, and rest of the variables are called the free variables.

So

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

are called the basis for the column space.

Q2. Given the  $3 \times 4$  matrix. Find the basis for Row space of A and its dimension

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

Soln: convert into Row echelon form

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

So the basis is  $\alpha_1 = [1, 4, 5, 2]$   $\alpha_2 = (0, 1, 1, 4/7)$

and the dimension for the row space of A is 2.

Q3. Determine the dimension of A and the Basis for Row and Column space of A.

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

Soln: convert into Row echelon form

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The Rank of A is 3 (No. of non-zero Rows)

So  $\dim(A) = 3$

Now Basis for Row space  $\alpha_1 = [1, 0, 1]$  ,  $\alpha_2 = [0, 1, -1]$

And another Basis for Row Space are original rows  
 corresponding to these rows

$$r_1 = [2 \ 0 \ 3] \quad , \quad r_2 = [1 \ 0 \ 1] \quad r_3 = [0 \ 2 \ -1]$$

Similarly

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad , \quad c_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad c_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

are the column spaces, and corresponding to the original matrix

$$e_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad , \quad e_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \quad , \quad e_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 4 \end{bmatrix}$$

Q. Find a basis for the null space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 2 & 6 & -5 & -2 \\ 0 & 0 & 5 & 10 \\ 2 & 6 & 0 & 8 \end{bmatrix}$$

Sol: Reduce the given matrix into row echelon form

$$\begin{pmatrix} 1 & 3 & -2 & 0 \\ 2 & 6 & -5 & -2 \\ 0 & 0 & 5 & 10 \\ 2 & 6 & 0 & 8 \end{pmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_4 \rightarrow R_4 - 2R_1 \end{array} \begin{pmatrix} 1 & 3 & -2 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 4 & 8 \end{pmatrix} \begin{array}{l} R_3 \rightarrow 5R_2 + R_3 \\ R_4 \rightarrow 5R_2 + R_4 \end{array}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now for the null space: The solution space of the homogeneous system of eqn  $AX=0$  which is subspace of  $\mathbb{R}^n$

$$\begin{pmatrix} 1 & 3 & -2 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

— leading variable (coz the coefficient of  $x$  is 1 & its leading in 1st row)

$$\textcircled{x} + 3y - 2z = 0 \quad \text{--- (1)}$$

$$\textcircled{-z} - 2w = 0 \quad \text{--- (2)}$$

leading variable

∴ Here free variables are  $y$  and  $w$

∴ dimension of null space is 2.

Since  $y$  &  $w$  are free so we can take any value of

(1) Let put  $y=1$  and  $w=0$ :

$$\begin{aligned} \text{from (2)} \quad & \cancel{x + 3x + 1 = z} \\ & -z - 2 \times 0 = 0 \\ & \Rightarrow z = 0 \end{aligned}$$

put in (1)

$$x + 3 \times 1 - 2 \times 0 = 0$$

$$x + 3 = 0$$

$$x = -3$$

∴  $(-3, 1, 0, 0)$  is the soln.

(2) Let  $y=0$  and  $w=1$ :

$$\begin{aligned} \text{from (2)} \quad & -z - 2 \times 1 = 0 \\ & -z = 2 \\ & z = -2 \end{aligned}$$

and

$$x + 0 + 4 = 0$$

$$x = -4$$

∴  $(-4, 0, -2, 0)$  is the soln.

Therefore

$\{(-3, 1, 0, 0), (-4, 0, -2, 0)\}$  is the basis for null space.

## Rank and Nullity:

(3)

Remark: The row space and column space of a matrix  $A$  have the same dimension.

Defn: The common dimension of the row and column space of a matrix  $A$  is called the rank of  $A$  and it is denoted by  $\text{rank}(A)$ . The dimension of the null space of  $A$  is called the nullity of  $A$  and is denoted by  $\text{nullity}(A)$ .

### Dimension Theorem for Matrix

If  $A$  is a matrix with  $n$  columns, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

Q1. Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Soln: Reduce this matrix to row echelon form

$$\begin{pmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & 1 & 2 & 12 & 16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Rank}(A) = 2 \quad (\text{No. of non zero rows})$$

Now By R. Nullity Theorem

$$\text{rank}(A) + \text{nullity}(A) = \text{No. of columns in the matrix}$$

$$2 + \text{nullity}(A) = 6$$

$$\text{nullity}(A) = 6 - 2 = 4$$