Week 2 - ch 1 math 251

system of Linear equations and matrices.

3 Ways we can solve the system of Equations

1. Grapically. 2. Algebracally.
3. Subtitution using algebracally

EX) $y = 3x - 4$ $y = -2x + 6$

find x and y?

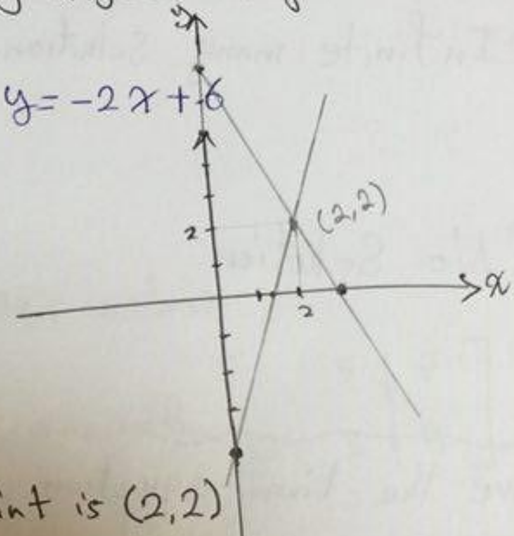
Sol.

eq①

x	0	$4/3$
y	-4	0

eq②

x	0	3
y	6	0



the intersection point is $(2,2)$

Now solve by algebracally:-

we can write $3x - y = 4$ $2x + y = 6$

$$\begin{array}{r} 3x - \cancel{y} = 4 \\ 2x + \cancel{y} = 6 \\ \hline 5x = 10 \end{array}$$

$\Rightarrow \boxed{x = 2}$

$2(2) + y = 6$

$y = 6 - 4 \Rightarrow \boxed{y = 2}$

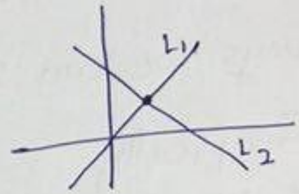
(x, y)

$(2, 2)$

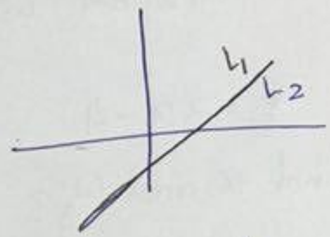
If we have two Lines L_1 and L_2 :-

three cases:-

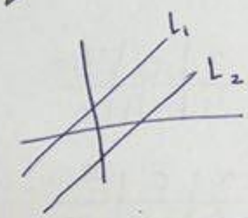
① Unique Solution



② Infinite many Solution.



③ No Solution



Ex) Solve the Linear equation

$$x - y = 1$$
$$2x + y = 6$$

Sol.

$$x - \cancel{y} = 1$$
$$2x + \cancel{y} = 6$$

$$3x = 7$$

$$\boxed{x = \frac{7}{3}}$$

$$\boxed{y = \frac{4}{3}}$$

$\left(\frac{7}{3}, \frac{4}{3}\right)$ is unique Solution.

Ex|

$$-3x - 3y = -12$$
$$3x + 3y = 6$$
$$0 + 0 = -6$$
$$0 = -6 \Rightarrow \text{No solution}$$

Gauss Elimination

Ex|

$$x_1 + 5x_2 = 7$$
$$-2x_1 - 7x_2 = -5$$

Sol. write the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ -2 & -7 & -5 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 3 & 9 \end{array}\right]$$

$$\xrightarrow{\frac{R_2}{3}} \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 \to R_1 - 5R_2} \left[\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1 & 3 \end{array}\right]$$

Row echelon form.

$$(x_1, x_2) = (-8, 3)$$

$$\boxed{\begin{array}{l} x_1 = -8 \\ x_2 = 3 \end{array}}$$

Ex] Solve the System

$$x - 3y + z = 4$$
$$2x - 8y + 8z = -2$$
$$-6x + 3y - 15z = 9$$

Sol. augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 6R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ 0 & -15 & -9 & 33 \end{array}\right]$$

$$\xrightarrow{\frac{R_2}{-2}} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & -15 & -9 & 33 \end{array}\right] R_3 \to R_3 + 15R_2$$

$$\xrightarrow[3R_2 + R_1]{15R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & -8 & 19 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & -54 & 108 \end{array}\right] \xrightarrow{\frac{R_3}{-54}} \left[\begin{array}{ccc|c} 1 & 0 & -8 & 19 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow[8R_3 + R_1]{3R_3 + R_2} \left[\begin{array}{c|c} \underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{\text{Identity matrix}} & \begin{array}{c} 3 \\ -1 \\ -2 \end{array} \end{array}\right]$$

The Solution of the System is $(x, y, z) = (3, -1, -2)$

Uniqe Solution

Only one Solution.

matrix

trace = Sum of the main diagonal.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ matrix مصفوفة

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ determinant محدد

Ex) find the $A - B^T + \frac{1}{2}C$ if

$$A = \begin{bmatrix} 7 & 12 \\ 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 1 \\ 10 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 14 & 6 \\ 8 & 2 \end{bmatrix}$$

Sol.

$$= \begin{bmatrix} 7 & 12 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 10 \\ 1 & 3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 14 & 6 \\ 8 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 12 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 10 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 7 & 3 \\ 4 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 7 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 4 & 2 \end{bmatrix} = \underline{L} \qquad \text{Trace } Tr(L) = 10 + 2 = 12$$

Sum of main diagonal

Ex) solve using Gauss Jordan :-

$$\left.\begin{aligned} x + 3y + z &= 10 \\ x - 2y - z &= -6 \\ 2x + y + 2z &= 10 \end{aligned}\right\} \rightarrow \text{system}$$

Sol.

augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 10 \\ 1 & -2 & -1 & -6 \\ 2 & 1 & 2 & 10 \end{array}\right] \xrightarrow[-2R_1 + R_3]{-R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 10 \\ 0 & -5 & -2 & -16 \\ 0 & -5 & 0 & -10 \end{array}\right]$$

replace R_2 and R_3

$$\xrightarrow{}\left[\begin{array}{ccc|c}1&3&1&10\\0&-5&0&-10\\0&-5&-2&-16\end{array}\right]\xrightarrow{\frac{R_2}{-5}}\left[\begin{array}{ccc|c}1&3&1&10\\0&1&0&2\\0&-5&-2&-16\end{array}\right]$$

$$\xrightarrow[5R_2+R_3]{-3R_2+R_1}\left[\begin{array}{ccc|c}1&0&1&4\\0&1&0&2\\0&0&-2&-6\end{array}\right]\xrightarrow{\frac{R_3}{-2}}\left[\begin{array}{ccc|c}1&0&1&4\\0&1&0&2\\0&0&1&3\end{array}\right]$$

$$\xrightarrow{-R_3+R_1}\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&2\\0&0&1&3\end{array}\right]$$

the Solution of the System is $(x,y,z)=(1,2,3)$

Ex) Solve using Gauss Jordan

$$\left.\begin{array}{l}x+y+z=4\\2x-3y+z=2\\-x+2y-z=-1\end{array}\right\}\text{system}$$

Sol.

$$\left[\begin{array}{ccc|c}1&1&1&4\\2&-3&1&2\\-1&2&-1&-1\end{array}\right]\xrightarrow[R_1+R_3]{-2R_1+R_2}\left[\begin{array}{ccc|c}1&1&1&4\\0&-5&-1&-6\\0&3&0&3\end{array}\right]$$

replace R_2 & R_3

$$\xrightarrow{}\left[\begin{array}{ccc|c}1&1&1&4\\0&3&0&3\\0&-5&-1&-6\end{array}\right]\xrightarrow{\frac{R_2}{3}}\left[\begin{array}{ccc|c}1&1&1&4\\0&1&0&1\\0&-5&-1&-6\end{array}\right]\xrightarrow[5R_2+R_3]{-R_2+R_1}\left[\begin{array}{ccc|c}1&0&1&3\\0&1&0&1\\0&0&-1&-1\end{array}\right]$$

$$\xrightarrow{R_3+R_1}\left[\begin{array}{ccc|c}1&0&0&2\\0&1&0&1\\0&0&-1&-1\end{array}\right]\xrightarrow{-R_3}\left[\begin{array}{ccc|c}1&0&0&2\\0&1&0&1\\0&0&1&1\end{array}\right]$$

the Solution of the System $(x,y,z)=(2,1,1)$.

math 251 - week 3 - ch1

types of Matrix:

① Zero matrix: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$ $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}_{1\times 3}$ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times 1}$

② Identity matrix:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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theorm 1.4.5: If the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad-bc \neq 0$, then the formula is

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Q.1) find the inverse of

ⓐ $A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$, ⓑ $B = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

Sol. ⓐ $ad-bc = 6\times 2 - 5\times 1 = 7 \neq 0$ has inventible exists.

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & \frac{-1}{7} \\ \frac{-5}{7} & \frac{6}{7} \end{bmatrix}$$

ⓑ $ad-bc = -1\times -6 - 3\times 2 = 6-6 = 0 = 0$ the inverse does not exist.

* Using the Row opertion to find the inverse.

Ex) find the inverse of $A = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix}$

Sol. write

$$[A|I] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -2 & -7 & 0 & 1 \end{array}\right] \; 2R_1 + R_2$$

convet to row reduced form (Identity matrix)

$$= \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & -1 \end{array}\right] \xrightarrow{-3R_2+R_1}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & 3 \\ 0 & 1 & -2 & -1 \end{array}\right] \Rightarrow$$

So we have $A^{-1} = \begin{bmatrix} 7 & 3 \\ -2 & -1 \end{bmatrix}$

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Ex) using the Row operation find A^{-1}

if $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Sol.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \xrightarrow[2R_2+R_3]{-2R_2+R_1}$$

week 3 - ch1

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\xrightarrow[-9R_3+R_1]{3R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

So, the inverse $\Rightarrow$ $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

Therom: 1.6.2 $\quad Ax = b \Rightarrow x = A^{-1}b$

Ex) find the solution of the system of linear equation using A^{-1}.

$$x + 3y = 1$$
$$2x + 5y = 3$$

Sol. we write $Ax = b$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \Rightarrow [A \mid I] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & +2 & -1 \end{array}\right] \xrightarrow{-3R_2+R_1}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

$$x = A^{-1} b \Rightarrow x = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} x = 4 \\ y = -1 \end{matrix}}$$

Ex) For what value $\underline{\underline{a}}$ and $\underline{\underline{b}}$, it the following system is inconsistent (No Solution), unique Solution, infinitly many solution.

$$x_1 + a x_2 = 1 \quad ; \quad x_1 + 2x_2 = b$$

Sol. write the augmented matrix

$$\left[\begin{array}{cc|c} 1 & a & 1 \\ 1 & 2 & b \end{array}\right] \xrightarrow{-R_1 + R_2} \left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & 2-a & b-1 \end{array}\right]$$

Case 1: $a = 2 \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & b-1 \end{array}\right]$

1. if $b = 1 \Rightarrow$ then Infinitely many Solution.
2. if $b \neq 1 \Rightarrow$ then No Solution.

se 2! $a \neq 2$

$$\left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & 2-a & b-1 \end{array}\right] \xrightarrow{\frac{R_2}{2-a}} \left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & 1 & \frac{b-1}{2-a} \end{array}\right]$$

unique Solution

Math 251 - week 4 - ch 2 Determinant

Minor and Cofactor: The minor of a_{ij} is denoted by M_{ij}.

The number $(-1)^{i+j} M_{ij}$ is denoted by C_{ij} is called cofactor.

Ex| Find the minor and cofactor

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

Sol. The minor of a_{11} is $M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 5\times 8 - 4\times 6 = \underline{\underline{16}}$

The cofactor of a_{11} is $C_{11} = (-1)^{1+1} M_{11} = -1^2 \times 16 = \underline{\underline{16}}$

The minor of a_{12} is $M_{12} = \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = \underline{\underline{10}}$

The cofactor of a_{12} is $C_{12} = (-1)^{1+2} M_{12} = -1^3 \times 10$

$= \underline{\underline{-10}}$

The minor of a_{32} is $M_{32} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 3\times 6 - 2(-4)$

$= \underline{\underline{26}}$

The cofactor of a_{32} is $C_{32} = (-1)^{3+2} M_{32}$

$= -1^5 \times 26 = \underline{\underline{-26}}$

Ex) find the determinant by cofactor expantion along first row then first column.

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

Sol. By the Row side

$$\det(A) = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3 \times (-4) - 1(-11) + 0$$

$$= -12 + 11 = \underline{\underline{-1}}$$

By the column side

$$\det(A) = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-4) + 2(-2-0) + 5(3-0)$$

$$= -12 - 4 + 15 = \underline{\underline{-1}}$$

$$\left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right]$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Find the value of determinant

$$A = \begin{bmatrix} 1 & 5 & -3 \\ 1 & 0 & 2 \\ 3 & -1 & 2 \end{bmatrix}$$

Sol.

$$A = \left[\begin{array}{ccc|cc} 1 & 5 & -3 & 1 & 5 \\ 1 & 0 & 2 & 1 & 0 \\ 3 & -1 & 2 & 3 & -1 \end{array}\right]$$

$$\det(A) = 1\times 0\times 2 + 5\times 2\times 3 + (-3)(1)(-1) - (-3)(0)(3) - (1)(2)(-1) - (5)(1)(2)$$

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③ Row Reduction Method

we convert the determinant into upper triangle matrix using Row operation.

Ex Evaluate $\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix}$

Sol. $\xrightarrow{R_2 \leftrightarrow R_1} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$ Pivot

$\xrightarrow{R_3 \to R_3 - 2R_1} = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \xrightarrow{R_3 \to R_3 - 10R_2}$

$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} = -3(-55)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix} = -3(-55)(1)$
$= 165$

Theorem 2.3

ⓐ $\det(B) = k \det(A)$

ⓑ $\det(B) = -\det(A)$

ⓒ $\det(A) = \det(B)$

مثال
Ex)

If $A = \begin{pmatrix} 5 & -2 & 1 \\ 0 & 3 & -1 \\ 2 & 0 & 7 \end{pmatrix}$ and $|A| = 103$

Evaluate the determinant of the following matrix

$B = \begin{pmatrix} 5 & -2 & 1 \\ 0 & 6 & -2 \\ 2 & 0 & 7 \end{pmatrix}$ $C = \begin{pmatrix} 5 & -2 & 1 \\ 2 & 0 & 7 \\ 0 & 3 & -1 \end{pmatrix}$

$D = \begin{pmatrix} 5 & -2 & 1 \\ 0 & 3 & -1 \\ 12 & -4 & 9 \end{pmatrix}$ $\underline{2R_1 + R_3}$ $2 \times 5 + 2 = 10 + 2 = 12$

Sol.

$|B| = 2|A| = 2 \times 103 = \underline{\underline{206}}$

$|C| = -|A| = \underline{\underline{-103}}$

$|D| = |A| = \underline{\underline{103}}$

Cramer's Rule

$$Ax = B$$

$$\det(A) \neq 0 \Rightarrow x_1 = \frac{\det(A_1)}{\det(A)}$$

$$x_2 = \frac{\det(A_3)}{\det(A)}, \quad x_3 = \frac{\det(A_3)}{\det(A)}$$

$$A_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad A_2 = \begin{vmatrix} a_{12} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad A_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Ex) Using Cramers's Rule to solve

$$x_1 + 2x_3 = 6$$

$$-3x_1 + 4x_2 + 6x_2 = 30$$

$$-x_1 - 2x_2 + 3x_3 = 8$$

Sol.

$$A = \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix}$$

$$A_1 = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix}, \quad A_2 = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix}$$

$$A_3 = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix}$$

$$|A| = 44$$

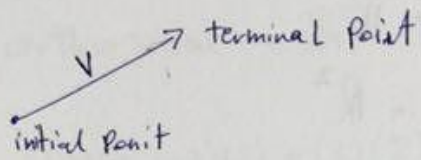
$$|A_1| = -40 \; ; \; |A_2| = 72 \; ; \; |A_3| = -152$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} =$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44}$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-152}{44}$$

Math 251 - week 5 - vectors



2D: (x_1, y_1)

3D: (x_1, y_1, z_1) (x_2, y_2, z_2)

V_1 V_2

Ex) if $V = (1, -3, 2)$ and $W = (4, 2, 1)$

find $V + W$ and $V - W$

sol.

Addition vectors $V + W = (5, -1, 3)$

Subtraction Vectors $V - W = (-3, -5, 1)$

Norm of a vector: $\|V\|$ ← length/distance of the Vector

in $\mathbb{R}^2$ $\|v\| = \sqrt{v_1^2 + v_2^2}$ 2D

in $\mathbb{R}^3$ $\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Ex) find the Norm of a vector

$V = (-3, 2, 1)$ in $\mathbb{R}^3$

Sol. $\|v\| = \sqrt{(-3)^2 + (2)^2 + (1)^2} = \sqrt{9+4+1} = \sqrt{14}$

distance of the vector

* **Unique Vector:** $u = \frac{1}{\|v\|} u$

the Standard unit vectors in $\mathbb{R}^2$

$i = (1, 0)$; $j = (0, 1)$

in $\mathbb{R}^3$ $i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$

Dot prodect: $u \cdot v = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$

or $\cos\theta = \frac{u \cdot v}{\|u\| \, \|v\|}$

Ex) Calculate the dot Product

$a = (1, 2, 3)$ $b = (4, -5, 6)$

Sol.

$a \cdot b = 1 \cdot 4 + 2 \cdot (-5) + 3 \cdot 6$

$= 4 - 10 + 18$

multiplication (product) $= 12$

Orthogonality: Two vectors are Perpendicular then $u \cdot v = 0$

Show that $u=(-2,3,1,4)$ and $v=(1,2,0,1)$ are orthogonal in R^4

Sol.

$$u \cdot v = (-2)(1) + (3)(2) + 1 \cdot 0 + 4 \cdot (-1)$$
$$= (-2 + 6 + 0 - 4)$$
$$= 6 - 6$$
$$= 0$$

So that means $\underline{u \perp v}$

Ex) if $a=(6,-1,3)$ for what value of C is the vector $b=(4,C,-2) \perp$ to $\vec{a}$

Sol.

$$a \cdot b = 6 \cdot 4 + (-1) \cdot C + 3 \cdot -2$$
$$a \cdot b = 24 - C - 6$$
$$a \cdot b = 18 - C$$
$$0 = 18 - C$$
$$\Rightarrow \boxed{C = 18}$$

So $\underline{a \perp b} \Rightarrow \underline{a \cdot b = 0}$

Cross Product: $u \times v$

if $u = (u_1, u_2, u_3)$ $\quad v = (v_1, v_2, v_3)$

Then

$$u \times v = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} =$$

$$u \times v = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

Ex) Calcullate the cross product of

$u = (1, 2, -2)$, $v = (3, 0, 1)$

Sol.

$$\begin{bmatrix} 1 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix} = \left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}, - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -2 \\ 3 & 0 \end{vmatrix} \right)$$

$$= (2, -7, -6)$$

Ex) find the cosine angle θ between the vectors

$u = (6, 3, 2)$ and $v = (2, 3, 6)$

Sol.

$$\cos \theta = \frac{u \cdot v}{\|u\| \, \|v\|}$$

$$u \cdot v = 6 \cdot 2 + 3 \cdot 3 + 2 \cdot 6 = 12 + 9 + 12 = 33$$

$$\|u\| = \sqrt{(6)^2+(3)^2+(2)^2} = \sqrt{49} = \underline{\underline{7}}$$

$$\|v\| = \sqrt{2^2+3^2+6^2} = \sqrt{49} = \underline{\underline{7}}$$

$$\Rightarrow \cos\theta = \frac{33}{7 \cdot 7} = \frac{33}{49}$$

Ex) Calculate $u\cdot(v \times w)$ where $u = (1, -3, 4)$ and $v = (2i - 3j + k)$, $w = (-i + 4j - 3k)$

Sol.

$$v \times w = \begin{bmatrix} i & j & k \\ 2 & -3 & 1 \\ -1 & 4 & -3 \end{bmatrix}$$

$$\Rightarrow \left(\underset{9-4}{i\begin{vmatrix} -3 & 1 \\ 4 & -3 \end{vmatrix}}, -j\begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix}, \underset{8-3}{k\begin{vmatrix} 2 & -3 \\ -1 & 4 \end{vmatrix}} \right)$$

$$\Rightarrow \left(i(5), j(5), k(5) \right) \Rightarrow v \times w = (5i, 5j, 5k)$$

$$u\cdot(v \times w) = (1, -3, 4)\cdot(5, 5, 5)$$

$$= (1.5 + (-3.5) + 4.5)$$

$$= (5 - 15 + 20)$$

$$u\cdot(v \times w) = 10$$

Ex) Show that $u=(2,1,0)$ and $V=(-3,6,5)$ are orthogonal.

Sol. $u.V=(2)(-3)+(1)(6)+(0)(5)$

$= -6+6+0$

$= 0$

$\Rightarrow u \perp V$

Theorm 3.34) $P(x_0, y_0)$

in R^2 $ax+by+c=0$

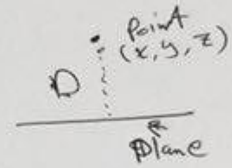
$$D=\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$$

in R^3

$$D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

Find the distance between the point $(\underset{x_0}{1}, \underset{y_0}{-4}, \underset{z_0}{-3})$ and a plane $\underset{a}{2}x - \underset{b}{3}y + \underset{c}{6}z + \underset{d}{1} = 0$

Sol.



$$D = \frac{|2.1 + (-3)(-4) + 6.(-3) + 1|}{\sqrt{2^2 + (-3)^2 + 6^2}}$$

$$\Rightarrow D = \frac{|-3|}{7} = \frac{3}{7}$$

math 251 - week 6 - ch 4 vector Space

n-space : R^n $(x_1, x_2, \dots, x_n)$

$n=1 \rightarrow R^1 = $ 1-Space Real numbers

$n=2 \rightarrow R^2 = $ 2-Spaces ordered reale numbers (x_1, x_2)

$n=3 \rightarrow R^3 = $ 3-Spaces ordered real numbers (x_1, x_2, x_3)

Vector space: $V \neq \phi$, addition and scaler multiplication $(V, +, \cdot)$

Addition

① $u + v \in V$

② $u + v = v + u$

③ $u + (v + w) = (u + v) + w$

④ $0 + u = u + 0 = u$

⑤ $u + (-u) = (-u) + u = 0$

Scaler multiplication

⑥ $ku \in V$

⑦ $k(u + v) = ku + kv$

⑧ $(k + m)u = ku + mu$

⑨ $k(mu) = (km)u$

⑩ $1 \cdot u = u$

① $V = \{0\}$ vector space

②

③ How to check the V.S. or Not.

① Axiom (1) and Axiom 6

then 2, 3, 4, 5, 7, 8, 9, 10

Ex) Zero is a vector space:

Sol. Define the addition and scaler multiplication

(1) $0 + 0 = 0 \in V$

(2) $k \cdot 0 = 0 \in V$

Ex) Set of all integers is not a V.S

Sol. let $1 \in V$, $\frac{1}{2} \in R$ $\rightarrow$ constant (scalar)

(1) $1 + 4 = 5 \in V$

(2) $\underbrace{\frac{1}{2}}_{\text{scalar}} \cdot \underbrace{1}_{\text{integer}} = \underbrace{\frac{1}{2}}_{\text{non-integer}} \notin V$ (Not integer)

Ex) Set of all second degree polynomial is not a vector space.

Sol. $p(x) = x^2$ $q(x) = -x^2 + x + 1$

(1) $p(x) + q(x) = \cancel{x^2} - \cancel{x^2} + x + 1 = \underbrace{x + 1}_{\text{first-degree}} \notin V$

(Not second degree)

• Subspace

(1) $u+v \in V$ $ku \in V$

Ex) Show that $U=\{(a,b,c) \mid \underbrace{a=b=c}_{\text{equal}}\}$ is subspace in R^3

Sol. $u=(a,a,a)$ $v=(b,b,b)$

(1) $u+v=(a,a,a)+(b,b,b)$
$=(a+b,a+b,a+b) \in U$

(2) $ku = k(a,a,a)=(ka,ka,ka) \in U$

Ex) W is a set of singular matrix of order 2
show that W is not subspace.

$\boxed{|A|=0}$ Singular matrix.

Sol. Let $A=\begin{bmatrix}1&0\\0&0\end{bmatrix} \in W$

$\boxed{|A|=0}$

$B=\begin{bmatrix}0&0\\0&1\end{bmatrix} \in W$
singular

(1) $A+B=\begin{bmatrix}1&0\\0&1\end{bmatrix} \notin W$

$\boxed{|A|=1\neq 0}$ not singular

it is not a subspace.

*Linear ~~comp~~ combination

$$V = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

Ex| find a linear combination:

let $u = (1, 2, -1)$ $v = (6, 4, 2)$

Show that $w = (9, 2, 7)$ is a L.C of u and v

Sol. $w = au + bv$ — ①

$(9, 2, 7) = a(1, 2, -1) + b(6, 4, 2)$

$= (a, 2a, -a) + (6b, 4b, 2b)$

$= (a + 6b, 2a + 4b, -a + 2b)$

$a + 6b = 9$ — ①
$2a + 4b = 2$ — ②
$-a + 2b = 7$ — ③

adding ② + ③

$\boxed{b = \frac{16}{8} = 2}$

$\boxed{a = -3}$

$W = au + bv$

بالتعويض

$W = -3u + 2v$

↳ linear combination

Linear independent (L.I) & Linear dependent (L.D)

$C_1 U_1 + C_2 U_2 + \cdots + C_n U_n = 0$

① $C_1 = C_2 = \cdots = C_n = 0$ L.I الإستقلال الخطي

② $C_1 \neq 0$ or any $C_n \neq 0$ L.D الإعتماد الخطي
$C_2 \neq 0$
$\vdots$
$C_n \neq 0$

Ex) the standard unit vectors in R^3 is L.I

Sol. $R^2 \Rightarrow (1,0), (0,1)$
$R^3 \Rightarrow (1,0,0), (0,1,0), (0,0,1)$

in R^3 $i = (1,0,0)$ $j = (0,1,0)$ $k = (0,0,1)$

$C_1 i + C_2 j + C_3 k = 0$

$C_1(1,0,0) + C_2(0,1,0) + C_3(0,0,1) = (0,0,0)$

$(C_1,0,0) + (0,C_2,0) + (0,0,C_3) = (0,0,0)$

$C_1 = 0$
$C_2 = 0$
$C_3 = 0$

$\Rightarrow S = \{i, j, k\}$ is L.I

Ex) $S = \{(1,-2,3)\ (5,6,-1)\ (3,2,1)\}$.

is L.I or L.D

Sol. first write linear combination

$$c_1(1,-2,3) + c_2(5,6,-1) + c_3(3,2,1) = 0$$

$$(c_1, -2c_1, 3c_1) + (5c_2, 6c_2, -c_2) + (3c_3, 2c_3, c_3) = (0,0,0)$$

$$(c_1 + 5c_2 + 3c_3,\ -2c_1 + 6c_2 + 2c_3,\ 3c_1 - c_2 + c_3) = (0,0,0)$$

$$\left.\begin{aligned} c_1 + 5c_2 + 3c_3 &= 0 \\ -2c_1 + 6c_2 + 2c_3 &= 0 \\ 3c_1 - c_2 + c_3 &= 0 \end{aligned}\right\}$$

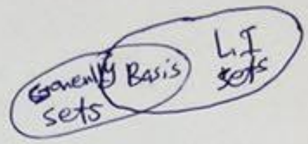
→ this system has nontrivial solution. الحل غير الصفري

hence, the vectors are Linearly dependent.

$$\det(A) = \begin{pmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{pmatrix} = 0 \quad ==> \quad \det(A) = 0$$

means this vectors are Linear Dependently

Basis & Dimension



① S is L.I

② S ~~span~~ spans V.

Dimension: $\dim(V)$

$\dim(\mathbb{R}^3)=3$, $\dim(\mathbb{R}^4)=4$

Row space, Column space and Null space:-

week 7

Ch4

Ex) find the Row and column vector of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

Sol. The Row vectors of A =

$r_1 = [\,2 \;\; 1 \;\; 0\,]$ $\quad r_2 = [\,3 \;\; -1 \;\; 4\,]$

$c_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\quad c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad c_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

Ex) find the basis for Row and column spaces of the matrix:

Leadings

Pivot

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Sol. ① Convert for row-echelon form.

Since the matrix is already in Row-echelon form therefore, The non-zero row vectors are

$r_1 = [\,1 \;\; -2 \;\; 5 \;\; 0 \;\; 3\,]$

$r_2 = [\,0 \;\; 1 \;\; 3 \;\; 0 \;\; 0\,]$

$r_3 = [\,0 \;\; 0 \;\; 0 \;\; 1 \;\; 0\,]$

these are called Basis for the Row Space.

$$C_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad C_2 = \begin{bmatrix}-2\\1\\0\\0\end{bmatrix} \quad C_4 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

are called the Basis for column Space.

Ex) Given 3x4 matrix find the basis for Row Spaces of A and its column

$$A = \begin{bmatrix}1 & 4 & 5 & 2\\ 2 & 1 & 3 & 0\\ -1 & 3 & 2 & 2\end{bmatrix}$$

Sol. Convert into Row-echelon form

$$A = \begin{bmatrix}1 & 4 & 5 & 2\\ 0 & 1 & 1 & 4/7\\ 0 & 0 & 0 & 7\end{bmatrix} \xrightarrow{\frac{R_3}{7}} \begin{bmatrix}1 & 4 & 5 & 2\\ 0 & 1 & 1 & 4/7\\ 0 & 0 & 0 & 1\end{bmatrix}$$

3-leadings

$r_1' = [1 \quad 4 \quad 5 \quad 2]$ $\qquad$ $r_1 = [1 \quad 4 \quad 5 \quad 2]$

$r_2' = [0 \quad 1 \quad 1 \quad 4/7]$ $\qquad$ $r_2 = [0 \quad 1 \quad 1 \quad 4/7]$

$r_3' = [0 \quad 0 \quad 0 \quad 1]$ $\qquad$ $r_3 = [0 \quad 0 \quad 0 \quad 7]$

Rank and Nullity

Rank Nullity Theorm:

$$\text{Rank}(A) + \text{nullity}(A) = n$$ no. of columns in the matrix

Ex) find the rank and nullity of the matrix:-

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Sol. After the Row echelon form:-

$$A = \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & 1 & 2 & 12 & 16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

$\text{Rank}(A) = 2$ (No. of non-zero rows)

$\dim(A) = 2$

$n = 6$ (No. of columns in the matrix)

$r(A) + \text{nullity}(A) = n$

$\Rightarrow 2 + \text{nullity}(A) = 6$

$\Rightarrow \text{nullity}(A) = 6 - 2 = 4$

Ex) find the dimension, Basis for row and column spaces.

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

Sol. Convert into Row-echelon form.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank(A) = 3

↓

(No. of non-zero rows)

~~dimen~~ dim(A) = 3

Row Spaces:-

$r'_1 = [1 \quad 0 \quad 1]$

$r'_2 = [0 \quad 1 \quad -1]$

$r'_3 = [0 \quad 0 \quad 1]$

OR

$r_1 = [2 \quad -1 \quad 3]$

$r_2 = [1 \quad 0 \quad 1]$

$r_3 = [0 \quad 2 \quad -1]$

Column Spaces:

$c'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad c'_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad c'_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

OR

$c_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad c_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \quad c_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 4 \end{bmatrix}$

Ex) find the rank and Nullity of the Matrix A.

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \end{bmatrix}$$

Sol. convert into Row-echelon form.

$$\xrightarrow[R_1+R_3]{-3R_1+R_2} A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & -12 \\ 0 & -7 & -1 & 12 \end{bmatrix} \xrightarrow{\frac{R_2}{7}} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1/7 & -12/7 \\ 0 & -7 & -1 & 12 \end{bmatrix}$$

$$\xrightarrow{7R_2+R_3} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1/7 & -12/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(A) = 2$

By the Rank Nullity Theorem:-

$$\text{Rank}(A) + \text{Nullity}(A) = n$$

$$2 + \text{Nullity}(A) = 4$$

$$\text{Nullity}(A) = 4 - 2 = 2$$