Example 1:

Let

$$\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ L_{21} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} \end{bmatrix}$$

then, comparing the left and right hand sides row by row implies that $U_{11} = 3$, $U_{12} = 1$, $L_{21}U_{11} = -6$ which implies $L_{21} = -2$ and $L_{21}U_{12} + U_{22} = -4$ which implies that $U_{22} = -2$. Hence

$$\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}$$

is an LU decomposition of $\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix}$.

Example 2:

Using material from the worked example in the notes we set

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix}$$

and comparing elements row by row we see that

$$\begin{array}{lll} U_{11} = 3, & U_{12} = 1, & U_{13} = 6, \\ L_{21} = -2, & U_{22} = 2, & U_{23} = -4 \\ L_{31} = 0 & L_{32} = 4 & U_{33} = -1 \end{array}$$

and it follows that

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

is an LU decomposition of the given matrix.

Example 3:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = LU$$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$.
Multiplying out LU and setting the answer equal to A gives

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}.$$

Now we have to use this to find the entries in L and U. Fortunately this is not nearly as hard as it might at first seem. We begin by running along the top row to see that

$$\boxed{U_{11} = 1}, \quad \boxed{U_{12} = 2}, \quad \boxed{U_{13} = 4}.$$

Now consider the second row

$$L_{21}U_{11} = 3 \quad \therefore L_{21} \times 1 = 3 \quad \therefore \boxed{L_{21} = 3},$$
$$L_{21}U_{12} + U_{22} = 8 \quad \therefore 3 \times 2 + U_{22} = 8 \quad \therefore \boxed{U_{22} = 2},$$
$$L_{21}U_{13} + U_{23} = 14 \quad \therefore 3 \times 4 + U_{23} = 14 \quad \therefore \boxed{U_{23} = 2}.$$

Notice how, at each step, the equation in hand has only one unknown in it, and other quantities that we have already found. This pattern continues on the last row

$$L_{31}U_{11} = 2 \quad \therefore L_{31} \times 1 = 2 \quad \therefore \boxed{L_{31} = 2},$$
$$L_{31}U_{12} + L_{32}U_{22} = 6 \quad \therefore 2 \times 2 + L_{32} \times 2 = 6 \quad \therefore \boxed{L_{32} = 1},$$
$$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 13 \quad \therefore (2 \times 4) + (1 \times 2) + U_{33} = 13 \quad \therefore \boxed{U_{33} = 3}.$$

We have shown that

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Example 4:

Let's see an example of LU-Decomposition without pivoting:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 12 \\ 0 & 2 & -10 \end{bmatrix}$$

The first step of Gaussian elimination is to subtract 2 times the first row form the second row. In order to record what was done, the multiplier, 2, into the place it was used to make a zero.

$$\xrightarrow{R2-2R1} \begin{bmatrix} 1 & -2 & 3 \\ (2) & -1 & 6 \\ 0 & 2 & -10 \end{bmatrix}$$

There is already a zero in the lower left corner, so we don't need to eliminate anything there. We record this fact with a (0). To eliminate a_{32}, we need to subtract -2 times the second row from the third row. Recording the -2:

$$\xrightarrow{R3-(-2)R2} \begin{bmatrix} 1 & -2 & 3 \\ (2) & -1 & 6 \\ (0) & (-2) & 2 \end{bmatrix}$$

Let U be the upper triangular matrix produced, and let L be the lower triangular matrix with the records and ones on the diagonal:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & -10 \end{bmatrix}$$

Then,

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 12 \\ 0 & 2 & -10 \end{bmatrix} = A$$