CHAPTER 1

ELEMENTARY ROW OPERATIONS —

(i) Multiplying a row by a non-zero constant.
(ii) Interchanging any two rows.
(iii) Adding a constant times one row to another row.

ROW-ECHELON FORM —

In any two successive rows that do not consist entirely of zeros, the leading '1' in lower row occurs farther to the right than the leading '1' in the higher row.

Examples
$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

REDUCED ROW-ECHELON FORM —

A row-echelon form is said to be Reduced Row Echelon form if each column of the matrix that contains a leading '1' has zeros everywhere else in that column.

Examples
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

GAUSS-ELIMINATION METHOD — The procedure (or algorithm) that reduces Augmented matrix (while solving a system of linear eqn.) to Row Echelon form is called Gaussian Elimination.

GAUSS-JORDAN ELIMINATION METHOD — The procedure (or algorithm) that reduces Augmented matrix to Reduced Row Echelon form is called Gauss-Jordan Elimination method.

SUM & DIFFERENCE OF MATRICES — If A & B are matrices of same size, then the Sum A+B is the matrix obtained by adding the entries of matrix B to the corresponding entries of matrix A.

The Difference A−B is the matrix obtained by subtracting the entries of matrix B from the corresponding entries of matrix A.

Matrices of different sizes cannot be added or subtracted.

SCALAR MULTIPLES OF MATRICES — If 'A' is any matrix and 'c' is any scalar, then the product 'cA' is the matrix obtained by multiplying each entry of matrix 'A' by constant c.

MULTIPLICATION OF MATRICES — If 'A' is a matrix of size $m \times r$ and 'B' is of size $r \times n$, then product 'AB' is a matrix of size $m \times n$ whose entries are determined as follows —

To find the entry in row i and column j of 'AB', single out row i from matrix 'A' and column j from matrix 'B'. Multiply the corresponding entries from the row and the column together and then add up the resulting products.

TRANSPOSE OF A MATRIX – If 'A' is any $m \times n$ matrix, then the Transpose of 'A' is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of 'A' and it is denoted by 'A^T'.

TRACE OF MATRIX – If 'A' is a square matrix, then the Trace of 'A', is defined to be the sum of the entries on the main diagonal of 'A' and is denoted by $tr(A)$. The trace of 'A' is undefined if 'A' is not a square matrix.

INVERSE OF A MATRIX: If 'A' is a square matrix, and if a matrix 'B' of the same size can be found such that $AB = BA = I$, then the matrix 'A' is said to be Invertible (or Non-Singular) and B is called an Inverse of 'A'.

If matrix 'A' is Invertible, then its inverse is denoted by A^{-1}

Thus, $$AA^{-1} = A^{-1}A = I$$

NOTE: The inverse of an Invertible matrix is Unique.

FORMULA FOR 2X2 MATRIX

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is Invertible iff $ad-bc \neq 0$ (ie., $|A| \neq 0$)

and $$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The quantity $ad-bc$ is called the Determinant of 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

ie., $$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$$

INVERSE OF MATRIX PRODUCTS

If A & B are invertible matrices with the same size, then AB is invertible

and $$(AB)^{-1} = B^{-1}A^{-1}$$

POWERS OF A MATRIX: If 'A' is a square matrix, then

$$A^0 = I \quad \text{and} \quad A^n = A.A.\cdots.A \quad [n \text{ factors}]$$

and if 'A' is invertible, then

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1}\cdots\cdots A^{-1} \quad [n \text{ factors}]$$

THEOREM: If 'A' is invertible and n is a non-negative integer, then

(i) A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(ii) A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

(iii) kA is invertible for any non-zero scalar k and $(kA)^{-1} = k^{-1}A^{-1}$.

Matrix Polynomials If 'A' is a square matrix, say $n \times n$ and if

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$ is any polynomial

then we define the $n \times n$ matrix $p(A)$ to be

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m \quad \text{——— (1)}$$

An expression of the form (1) is called a Matrix Polynomial in 'A'.

PROPERTIES OF TRANSPOSE

THEOREM (1) If the sizes of the matrices are such that the stated operations can be performed, then —

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(kA)^T = kA^T$

(iv) $(AB)^T = B^T A^T$

THEOREM (2) If 'A' is invertible matrix, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

ROW EQUIVALENT MATRICES:- Matrices A & B are said to be Row Equivalent if either (hence each) can be obtained from the other by a sequence of elementary row operations.

ELEMENTARY MATRIX:- An $n \times n$ matrix is called an Elementary Matrix if it can be obtained from $n \times n$ identity matrix I_n by performing a single elementary row operation.

THEOREM: If 'A' is an $n \times n$ matrix, then the following statements are equivalent —

(i) 'A' is invertible.

(ii) $Ax = 0$ has only trivial solution (or zero soln.)

(iii) The reduced row echelon form of A is I_n.

(iv) 'A' is expressible as a product of elementary matrices.

INVERSION ALGORITHM – To find the Inverse of an invertible matrix A, find a sequence of elementary row operations that reduces 'A' to the Identity matrix and then perform the same sequence of operations on I_n to obtain A^{-1}.

SOLVING LINEAR SYSTEMS BY MATRIX INVERSION

If 'A' is an invertible matrix of size $n \times n$, then for each $n \times 1$ matrix b, the system of equ. $Ax = b$ has exactly one solu., namely, $x = A^{-1}b$.

Note – This method only applies when the system has as many equations as unknowns and the coefficient matrix is Invertible.

DIAGONAL MATRICES – A square matrix in which all entries except the main diagonal are zero, is called Diagonal matrix.

Inverse of Diagonal Matrix –

A Diagonal matrix is Invertible iff all of its diagonal entries are non-zero.

Power of Diagonal Matrix –

TRIANGULAR MATRICES. – A matrix that is either Upper-triangular or lower-triangular is called the Triangular Matrix.

Note ① The Diagonal matrices are both Upper triangular and Lower triangular.

Note ② A triangular matrix is Invertible iff its diagonal entries are all non-zero.

PROPERTIES OF TRIANGULAR MATRICES.

① The Transpose of a Lower Triangular matrix is Upper Triangular Matrix and The Transpose of an Upper Triangular Matrix is Lower Triangular Matrix.

② The Product of Lower Triangular Matrices is Lower Triangular Matrix and The Product of Upper Triangular Matrices is Upper Triangular Matrix.

③ The Inverse of an invertible Lower Triangular Matrix is Lower Triangular and The Inverse of an invertible Upper Triangular Matrix is Upper Triangular.

SYMMETRIC MATRICES. A square matrix 'A' is said to be Symmetric if $A^T = A$.

PROPERTIES: If A & B are symmetric matrices with same size and k is any scalar, then

i) A^T is Symmetric (ii) $A+B$ and $A-B$ are symmetric (iii) kA is symmetric.

NOTE – The Product of two symmetric matrices is symmetric iff the matrices commute.

INVERTIBILITY OF SYMMETRIC MATRICES – In general, a symmetric matrix need not be Invertible For example, a diagonal matrix with a zero on main diagonal is symmetric but not Invertible.

THEOREM – If 'A' is an invertible symmetric matrix, then A^{-1} is symmetric.

NOTE – The Products AA^T and A^TA are always Symmetric.

THEOREM: If 'A' is an invertible matrix, then AA^T and A^TA are also invertible.

CHAPTER (2)

MINORS AND COFACTORS.

If 'A' is a square matrix, then the Minor of entry a_{ij} is defined to be the determinant of the submatrix that remains after deleting the ith row and jth column from matrix 'A', and is denoted by M_{ij}.

The no. $(-1)^{i+j} M_{ij}$ is called the cofactor of entry a_{ij} and is denoted by C_{ij}.

DETERMINANT OF A MATRIX – If 'A' is an $n \times n$ matrix, then the no. obtained by multiplying the entries in any row or column of 'A' by the corresponding cofactors and adding the resulting products is called the Determinant of matrix 'A'. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \quad \text{[Cofactor expansion along jth column]}$$

or

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \quad \text{[Cofactor expansion along ith row]}$$

Determinant of a Triangular Matrix –

If 'A' is an $n \times n$ Triangular matrix (upper triangular, lower triangular or diagonal), then det(A) is the product of entries on the main diagonal of the matrix, that is,

$$\det(A) = a_{11}\, a_{22} \cdots a_{nn}$$

THEOREM: Let 'A' be a square matrix. If 'A' has a row of zeros or a column of zeros, then $\det(A) = 0$.

THEOREM: Let 'A' be a square matrix, then $\det(A) = \det(A^T)$.

THEOREM: Let 'A' is an $n \times n$ matrix, then

(i) If 'B' is the matrix that results when a single row or single column of 'A' is multiplied by a scalar k, then $\det(B) = k\det(A)$.

(ii) If 'B' is the matrix that results when two rows or two columns of 'A' are interchanged, then $\det(B) = -\det(A)$

(iii) If 'B' is the matrix that results when a multiple of one row of 'A' is added to another row or when a multiple of one column is added to another column, then $\det(B) = \det(A)$.

THEOREM: If 'A' is a square matrix with ~~two~~ proportional rows or two proportional columns, then $\det(A) = 0$.

BASIC PROPERTIES OF DETERMINANTS.

① Let 'A' is an $n \times n$ matrix and k is any scalar, then $\det(kA) = k^n \det(A)$.

② Let A & B are $n \times n$ matrix, then usually $\det(A+B) \neq \det(A) + \det(B)$.

③ If A & B are square matrices of same size, then $\det(AB) = \det(A) . \det(B)$.

DETERMINANT TEST FOR INVERTIBILITY.

THEOREM: A square matrix 'A' is Invertible if and only if $\det(A) \neq 0$.

NOTE: If a matrix 'A' is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

ADJOINT OF A MATRIX – The transpose of the matrix formed by the cofactors of matrix, is called Adjoint of matrix.

If 'A' is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then

$$adj(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$$

INVERSE OF A MATRIX USING ITS ADJOINT

If 'A' is an invertible matrix, then $A^{-1} = \dfrac{1}{\det(A)}[adj(A)]$

CRAMER'S RULE: If $Ax = b$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution given by

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad - - - - , \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing entries in j^{th} column of 'A' by the entries in the matrix b, where $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

CHAPTER 3

VECTORS WHOSE INITIAL POINT IS NOT AT THE ORIGIN

If $\overrightarrow{P_1P_2}$ denotes the vector with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then the components of this vector are given by

$$\overrightarrow{P_1P_2} = (x_2 - x_1,\ y_2 - y_1)$$

The components of a vector in 3-space that has initial point $P_1(x_1, y_1, z_1)$ and the terminal point $P_2(x_2, y_2, z_2)$ are given by $\overrightarrow{P_1P_2} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$

n-SPACE – If n is a positive integer, then an Ordered n-tuple is a sequence of n real numbers $(v_1, v_2, \ldots, v_n)$. The set of all ordered n-tuples is called the n-Space and is denoted by R^n.

Operations on Vectors in R^n

If $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n and if k is any scalar, then we define

$$\vec{u} + \vec{v} = (u_1 + v_1,\ u_2 + v_2,\ \ldots,\ u_n + v_n)$$
$$k\vec{u} = (ku_1,\ ku_2,\ \ldots,\ ku_n)$$
$$-\vec{u} = (-u_1, -u_2, \ldots, -u_n)$$
$$\vec{v} - \vec{u} = (v_1 - u_1,\ v_2 - u_2,\ \ldots,\ v_n - u_n)$$

LINEAR COMBINATIONS. If $\vec{w}$ is a vector in R^n, then $\vec{w}$ is said to be a Linear Combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ in R^n if it can be expressed in the form

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r$$

NORM OF A VECTOR (OR Length OR Magnitude)

The Norm of a vector (v_1, v_2) in R^2 is $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$

The Norm of a vector (v_1, v_2, v_3) in R^3 is $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

In general, the Norm of vector $(v_1, v_2, \ldots, v_n)$ in R^n is $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

NOTE: $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0$ iff $\vec{v} = 0$.

UNIT VECTORS – A vector of norm '1' is called a Unit Vector.

If $\vec{v}$ is any non-zero vector in R^n, then $\hat{u} = \dfrac{1}{\|\vec{v}\|}\vec{v}$ defines a Unit Vector in the same direction as the vector $\vec{v}$.

THE STANDARD UNIT VECTORS

The standard unit vectors in R^2 are denoted by $\hat{i} = (1,0)$ & $\hat{j} = (0,1)$; the standard unit vectors in R^3 are denoted by $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$ & $\hat{k} = (0,0,1)$.

DISTANCE IN R^n:

The distance between the points $P_1(x_1, y_1)$ & $P_2(x_2, y_2)$ in 2-space is

$$d = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

and the distance between the points $P_1(x_1, y_1, z_1)$ & $P_2(x_2, y_2, z_2)$ in 3-space is

$$d = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

DOT PRODUCT: If $\vec{u}$ & $\vec{v}$ are non-zero vectors in R^2 or R^3 and θ is the angle between $\vec{u}$ and $\vec{v}$ then the Dot product of $\vec{u}$ and $\vec{v}$ is defined as

$$\vec{u}.\vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta \quad , \quad \text{where } 0 \le \theta \le \pi .$$

and so $$\cos\theta = \frac{\vec{u}.\vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

COMPONENT FORM OF DOT PRODUCT –

If $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the Dot Product (also called Euclidean Inner Product) of $\vec{u}$ & $\vec{v}$ is defined as

$$\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

ALGEBRAIC PROPERTIES OF DOT PRODUCT –

If $\vec{u}, \vec{v}$ & $\vec{w}$ are vectors in R^n and k is a scalar, then

(i) $\vec{u}.\vec{v} = \vec{v}.\vec{u}$ (Symmetric Property)

(ii) $\vec{u}.(\vec{v}+\vec{w}) = \vec{u}.\vec{v} + \vec{u}.\vec{w}$ (Distributive Property)

(iii) $k(\vec{u}.\vec{v}) = (k\vec{u}).\vec{v}$ (Homogeneity Property)

(iv) $\vec{v}.\vec{v} \ge 0$ and $\vec{v}.\vec{v} = 0$ iff $\vec{v} = 0$ (Positivity Property)

NOTE: If $\vec{v} = (v_1, v_2, \ldots, v_n)$, then

$$\vec{v}.\vec{v} = v_1^2 + v_2^2 + \cdots + v_n^2$$

$$\vec{v}.\vec{v} = \|\vec{v}\|^2 \quad \text{OR} \quad \|\vec{v}\| = \sqrt{\vec{v}.\vec{v}}$$

ORTHOGONAL VECTORS – Two non-zero vectors $\vec{u}$ & $\vec{v}$ in R^n are said to be Orthogonal or Perpendicular (i.e., $\theta = \frac{\pi}{2}$) if $\vec{u}.\vec{v} = 0$.

A non-empty set of vectors in R^n is called an Orthogonal Set if all pairs of distinct vectors in the set are Orthogonal.

An orthogonal set of unit vectors is called an Orthonormal Set.

ORTHOGONAL PROJECTIONS

Vector Component of a vector $\vec{u}$ along the vector $\vec{a}$ is given by

$$\vec{w}_1 = \text{proj}_{\vec{a}}\vec{u} = \left(\frac{\vec{u}.\vec{a}}{\|\vec{a}\|^2}\right)\vec{a}$$

and Vector Component of vector $\vec{u}$ orthogonal to $\vec{a}$ is $W_2 = \vec{u} - \text{proj}_{\vec{a}}\vec{u}$

THEOREM OF PYTHAGORAS IN R^n: If $\vec{u}$ & $\vec{v}$ are orthogonal vectors in R^n with Euclidean inner product, then $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

NOTE: In R^2, the Distance between the point $P_0(x_0, y_0)$ and the line $ax+by+c=0$ is given by

$$D = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$$

CROSS-PRODUCT OF VECTORS: If $\vec{u} = (u_1, u_2, u_3)$ & $\vec{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the cross-product $\vec{u}\times\vec{v}$ is the vector defined by

$$\vec{u}\times\vec{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \qquad \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

RELATIONSHIP INVOLVING CROSS PRODUCT AND DOT PRODUCT

If $\vec{u}, \vec{v}$ & $\vec{w}$ are vectors in 3-space, then

(i) $\vec{u}.(\vec{u}\times\vec{v}) = 0$ ($\vec{u}\times\vec{v}$ is orthogonal to $\vec{u}$)

(ii) $\vec{v}.(\vec{u}\times\vec{v}) = 0$ ($\vec{u}\times\vec{v}$ is orthogonal to $\vec{v}$)

(iii) $\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}.\vec{v})^2$ [Lagrange's Identity]

PROPERTIES OF CROSS PRODUCT –

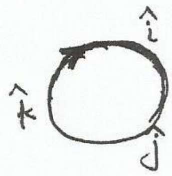
If $\vec{u}, \vec{v}$ & $\vec{w}$ are any vectors in 3-space and k is any scalar, then

(i) $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$

(ii) $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$

(iii) $k(\vec{u}\times\vec{v}) = (k\vec{u})\times\vec{v} = \vec{u}\times(k\vec{v})$

(iv) $\vec{u}\times\vec{u} = 0$



STANDARD UNIT VECTORS:-

$\hat{i}\times\hat{i} = 0$ $\quad \hat{j}\times\hat{j} = 0$ $\quad \hat{k}\times\hat{k} = 0$

$\hat{i}\times\hat{j} = \hat{k}$ $\quad \hat{j}\times\hat{k} = \hat{i}$ $\quad \hat{k}\times\hat{i} = \hat{j}$

$\hat{j}\times\hat{i} = -\hat{k}$ $\quad \hat{k}\times\hat{j} = -\hat{i}$ $\quad \hat{i}\times\hat{k} = -\hat{j}$

DETERMINANT FORM OF CROSS-PRODUCT –

If $\vec{u} = (u_1, u_2, u_3)$ & $\vec{v} = (v_1, v_2, v_3)$ are vectors in R^3, then $\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_3 & v_3 \end{vmatrix}$

GEOMETRIC INTERPRETATION OF CROSS-PRODUCT

If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\|\vec{u}\times\vec{v}\|$ is equal to the Area of the Parallelogram determined by $\vec{u}$ & $\vec{v}$.

CHAPTER (4)

VECTOR SPACE: Let V be an arbitrary non-empty set of objects on which two operations are defined — Addition and Multiplication by Scalars.

If 10 axioms are satisfied by all objects $u, v, \& w$ in V and all scalars $k \& m$, then we call V, a vector space and we call objects in V as vectors.

Example (1) The Zero Vector Space

Example (2) R^n is a vector space

Example (3) The set of all $m \times n$ matrices i.e., M_{mn} is a Vector space

Example (4) The set of real-valued functions i.e., $F(-\infty, \infty)$ is a Vector Space

SUBSPACES: A subset W of a vector space V is called a Subspace of V if W is itself a vector space under Addition and Scalar Multiplication defined on V.

THEOREM: If W is a set of one or more vectors in a vector space V, then W is a Subspace of V iff the following conditions hold —

(i) If $u, v \in W$, then $u+v \in W$.

(ii) If $u \in W$ and k is any scalar, then $ku \in W$.

NOTE (1) Every vector space has at least two Subspaces, itself and its zero subspace.

NOTE (2) Lines through the Origin are Subspaces of R^2 and of R^3.

NOTE (3) Planes through the Origin are Subspaces of R^3.

THEOREM: If $W_1, W_2, \cdots, W_r$ are the subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.

~~THEOREM: If S = W1, W2~~

THEOREM: If $S = \{w_1, w_2, \cdots, w_r\}$ is a non-empty set of vectors in a vector space V, then (i) The set W of all possible linear combination of vectors in S is a Subspace of V.

(ii) The set W in part (i) is the 'smallest' subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains W.

THE SPAN OF S

The subspace of a vector space V that is formed from all possible linear combinations of the vectors in a non-empty set S is called the Span of S, and we say that the vectors in S span that subspace.

If $S = \{w_1, w_2, \cdots, w_r\}$, then we denote Span of S by $\text{span}\{w_1, w_2, \cdots, w_r\}$ or $\text{span}(S)$.

LINEAR INDEPENDENCE: If $S = \{v_1, v_2, \ldots, v_r\}$ is a non-empty set of vectors in a vector space V, then the vector equ.

$$k_1 v_1 + k_2 v_2 + \cdots\cdots + k_r v_r = 0$$

has at least one solu., namely $k_1 = 0, k_2 = 0, \ldots, k_r = 0$.
If this is the only solu. (called trivial solu.), then S is said to be Linearly Independent.
If there are solutions in addition to trivial solu., then S is said to be Linearly Dependent Set.

NOTE: If $|A| \neq 0$, then the vectors are Linearly Independent,
and if $|A| = 0$, then the vectors are Linearly Dependent.

SETS WITH ONE OR TWO VECTORS.

THEOREM (i) A finite set that contains '0' is Linearly Dependent.
(ii) A set with exactly one vector is Linearly Independent iff that vector is not '0'.
(iii) A set with exactly two vectors is Linearly Independent iff neither vector is a scalar multiple of the other.

BASIS FOR A VECTOR SPACE: If V is any vector space and $S = \{v_1, v_2, \ldots, v_n\}$ is a finite set of vectors in V, then S is called a Basis for V if the following two conditions hold —

(i) The set S is Linearly Independent. (i.e. The vectors $v_1, v_2, \ldots, v_n$ are L.I.)
(ii) The set S spans V. (i.e. Every vector in V can be expressed as a linear combination of $v_1, v_2, \ldots, v_n$)

NOTE ① The Set $S = \{\hat{i}, \hat{j}\}$ is the standard basis for R^2,
where $\hat{i} = (1,0)$ and $\hat{j} = (0,1)$.
and $S = \{\hat{i}, \hat{j}, \hat{k}\}$ is the standard basis for R^3,
where $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$ and $\hat{k} = (0,0,1)$.

NOTE ② $S = \{1, x, x^2, \ldots, x^n\}$ is the standard basis for the vector space P_n of polynomials of degree n or less.
In particular, the standard basis for P_2 is $\{1, x, x^2\}$
and the standard basis for P_3 is $\{1, x, x^2, x^3\}$.

CO-ORDINATES RELATIVE TO A BASIS.

If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, and

$$\mathbf{v} = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars $c_1, c_2, \ldots, c_n$ are called the Co-ordinates of $\mathbf{v}$ relative to the basis S.

We can write, $(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$ OR $[\mathbf{v}]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

No. of Vectors in a Basis

THEOREM: All bases for a finite-dimensional vector space have the same no. of vectors.

THEOREM: Let V is a finite-dimensional vector space and let $\{v_1, v_2, \dots, v_n\}$ is any basis.

(i) If a set has more than n vectors, then it is Linearly Dependent.

(ii) If a set has fewer than n vectors, then it does not span V.

DIMENSION OF VECTOR SPACE: The Dimension of a finite-dimensional vector space V is defined to be the no. of vectors in a basis for V and is denoted by $\dim(V)$.

In addition, the zero vector space is defined to have dimension zero.

NOTE:

$\dim(R^n) = n$ The standard basis has n-vectors.

$\dim(P_n) = n+1$ The standard basis has $(n+1)$ vectors.

$\dim(M_{mn}) = mn$ The standard basis has mn vectors.

ROW SPACE, COLUMN SPACE AND NULL SPACE

If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the Row Space of A and the subspace of R^m spanned by the column vectors of A is called the Column Space of A.

The solution space of the homogeneous system of equations $Ax = 0$ (which is a subspace of R^n), is called the Null Space of A.

THEOREM: A system of linear equations $Ax = b$ is consistent if and only if 'b' is in the column space of 'A'.

THEOREM: Elementary row operations do not change the Null space of a matrix.

THEOREM: Elementary row operations do not change the Row Space of a matrix.

THEOREM: If a matrix R is in Row Echelon form, then the row vectors with the leading '1's (the non-zero row vectors) form a basis for the row space of R and the column vectors with leading 1's of row vectors form a basis for column space of R.

DIMENSION OF ROW SPACE: The dimension of row space is the no. of basis vectors for the row space of matrix A, thus the dimension of row space is the no. of non-zero rows in Echelon form of A.

THEOREM: The Row Space and Column space of a matrix A have same dimension.

RANK AND NULLITY OF A MATRIX:

The common dimension of the Row Space and Column space of a matrix 'A' is called the Rank of 'A' and is denoted by rank(A); the dimension of the null space of 'A' is called the Nullity of 'A' and is denoted by nullity(A).

NOTE① The rank of 'A' can be interpreted as the no. of leading 1's in any row echelon form of the matrix 'A'.

NOTE② If 'A' is any $m \times n$ matrix, then $\text{rank}(A) \leq \min(m, n)$.

DIMENSION THEOREM FOR MATRICES.

If 'A' is a matrix with n columns, then $\text{rank}(A) + \text{nullity}(A) = n$.

THEOREM. If 'A' is an $m \times n$ matrix, then

(i) $\text{rank}(A)$ = the no. of leading variables in the general soln. of $Ax = 0$.

(ii) $\text{nullity}(A)$ = the no. of parameters in the general soln. of $Ax = 0$.

FUNDAMENTAL SPACES OF A MATRIX

If 'A' is an $m \times n$ matrix, then the row space and null space of 'A' are the subspaces of R^n and the column space of 'A' and null space of 'A^T' are the subspaces of R^m.

These spaces are called Fundamental Spaces of a matrix 'A'.

THEOREM. If 'A' is any matrix, then $\text{rank}(A) = \text{rank}(A^T)$.

QUESTIONS FROM CHAPTERS ① to ④

Q① Determine whether the following system has no solu., exactly one solu. or infinitely many solutions —

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$

Q② Solve the following system of linear equations by Gauss-Elimination method —

$$\begin{aligned} x + y + 2z &= 8 \\ -x - 2y + 3z &= 1 \\ 3x - 7y + 4z &= 10 \end{aligned}$$

Q③ Verify $(AB)^{-1} = B^{-1}A^{-1}$ for the matrices $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$.

Q④ Using Row operations, find the inverse of matrix, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.
Also find the inverse using $\operatorname{adj}(A)$.

Q⑤ Verify that AA^T is symmetric for the matrix $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$.

Q⑥ Find the determinant of the matrix, $A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

by cofactor expansion along the first column of A. Also find the determinant of matrix along first row of A.

Q⑦ Use Cramer's Rule to solve the following system

$$\begin{aligned} x \phantom{{}+4y} + 2z &= 6 \\ -3x + 4y + 6z &= 30 \\ -x - 2y + 3z &= 8 \end{aligned}$$

Q⑧ Find the Unit vector $\vec{u}$ that has same direction as $\vec{v} = (2, 2, -1)$.

Q⑨ If $\vec{u} = (1, -2, 2)$, $\vec{v} = (-3, 0, 4)$ and $\vec{w} = (6, -3, -2)$, then find $2\|\vec{u}\| - \|\vec{v}\| + \|3\vec{w}\|$.

Q⑩ Show that the set $S = \{\hat{i}, \hat{j}, \hat{k}\}$ of standard unit vectors is an orthogonal set in R^3.

Q⑪ Find cross product $\vec{u} \times \vec{v}$, where $\vec{u} = (1, 2, -2)$ and $\vec{v} = (3, 0, 1)$.

Q⑫ Find a vector that is orthogonal to both $\vec{u} = (1, 2, -2)$ and $\vec{v} = (3, 0, 1)$.

Q(13) Determine whether $v_1 = (1,1,2)$, $v_2 = (1,0,1)$ and $v_3 = (2,1,3)$ span the vector space R^3.

Q(14) Determine whether the vectors $v_1 = (1,-2,3)$, $v_2 = (5,6,-1)$, $v_3 = (3,2,1)$ are linearly independent or dependent in R^3.

Q(15) Find the coordinate vector of $W = (7,5)$ relative to the basis $\{u_1 = (2,-4),\ u_2 = (3,8)\}$ for R^2.

Q(16) Find the Dimension of the Row space of $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

Q(17) If $T(x_1, x_2) = (x_1 - x_2,\ 2x_1,\ 3x_2 + x_1)$, then find —
(i) the domain of T (ii) the codomain of T (iii) the image of $(1,-2)$.