SEC (8.1) GENERAL LINEAR TRANSFORMATIONS

Upto now our study of linear transformations has focused on transformations from R^n to R^m. In this Section, we will turn our attention to linear transformations involving general vector spaces. We will illustrate ways in which such transformations arise, and we will establish a fundamental relationship between general n-dimensional vector spaces and R^n.

Definitions and Terminology :

In Section 4.9 we defined a Matrix Transformation $T_A : R^n \to R^m$ to be a mapping of the form

$$T_A(\mathbf{x}) = A\mathbf{x}$$

in which A is an $m \times n$ matrix. We subsequently established that the matrix transformations are precisely the Linear Transformations from R^n to R^m, that is, the transformations with the linearity properties

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

$$\text{and} \quad T(k\mathbf{u}) = kT(\mathbf{u})$$

We will use these two properties for defining more general linear transformations —

DEFINITION

If $T: V \to W$ is a function from a vector space V to a vector space W, then T is called a Linear Transformation from V to W if the following two properties hold for all vectors $\mathbf{u}$ & $\mathbf{v}$ in V and for all scalars k —

(i) $T(k\mathbf{u}) = kT(\mathbf{u})$ [Homogeneity Property]

(ii) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ [Additivity Property]

In the special case where $V = W$, the linear transformation T is called a Linear Operator on the vector space V.

NOTE : The Homogeneity and Additivity properties of a linear transformation $T: V \to W$ can be used in combination to show that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors in V and k_1, k_2 are scalars, then

$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2) = k_1 T(\mathbf{v}_1) + k_2 T(\mathbf{v}_2)$$

More generally, if $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ are vectors in V and $k_1, k_2, \cdots, k_r$ are any scalars, then

$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r) = k_1 T(\mathbf{v}_1) + k_2 T(\mathbf{v}_2) + \cdots + k_r T(\mathbf{v}_r) \quad —(1)$$

THEOREM (1) If $T: V \to W$ is a linear transformation, then

(i) $T(\mathbf{0}) = \mathbf{0}$.

(ii) $T(\mathbf{u}-\mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all $\mathbf{u}$ & $\mathbf{v}$ in V.

Example ① Matrix Transformations

Because we have based the defi. of a general linear transformation on the Homogeneity & Additivity properties of Matrix Transformations, it follows that a matrix transformation $T_A : R^n \to R^m$ is also a Linear Transformation in this more general sense with $V = R^n$ & $W = R^m$.

Example ② The Zero Transformation

Let V & W are any two vector spaces. The mapping $T: V \to W$ such that $T(\mathbf{v}) = \mathbf{0}, \forall \mathbf{v} \in V$ is a linear transformation, called the Zero Transformation.

To see that T is linear, observe that

$$T(u+v) = 0, \quad T(u) = 0, \quad T(v) = 0 \quad \& \ T(ku) = 0, \text{ by defi.}$$

Therefore,
$$T(u+v) = T(u) + T(v)$$
and
$$T(ku) = k\,T(u)$$

Example ③ The Identity Operator

Let V is any vector space. The mapping $I : V \to V$ defined by $I(\mathbf{v}) = \mathbf{v}$ is called the Identity Operator on V. We can verify that I is linear.

Example ④ Dilation and Contraction Operators

If V is a vector space and k is any scalar, then the mapping $T: V \to V$ given by $T(\mathbf{x}) = k\mathbf{x}$ is a linear operator on V.

Because if c is any scalar and if $\mathbf{u}$ & $\mathbf{v}$ are any vectors in V, then

$$\begin{aligned} T(c\mathbf{u}) &= k(c\mathbf{u}) \\ &= c(k\mathbf{u}) \\ \Rightarrow \quad T(c\mathbf{u}) &= c\,T(\mathbf{u}) \end{aligned}$$

and
$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= k(\mathbf{u}+\mathbf{v}) \\ &= k\mathbf{u} + k\mathbf{v} \\ T(\mathbf{u}+\mathbf{v}) &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

If $0 < k < 1$, then T defined above is called the Contraction of V with factor k, and if $k > 1$, it is called the Dilation of V with factor k.

Example ⑤ A Linear Transformation from P_n to P_{n+1}

Let $p = p(x) = c_0 + c_1 x + \cdots + c_n x^n$ be a polynomial in P_n, then the transformation $T: P_n \to P_{n+1}$ defined by $T(p) = T(p(x)) = x p(x) = c_0 x + c_1 x^2 + \cdots + c_n x^{n+1}$ is a linear transformation.

Because for any scalar k and any polynomials p_1 & p_2 in P_n, we have

$$\begin{aligned} T(kp) &= T(k\, p(x)) \\ &= x[k p(x)] \\ &= k[x p(x)] \\ T(kp) &= k T(p) \end{aligned}$$

and

$$\begin{aligned} T(p_1 + p_2) &= T(p_1(x) + p_2(x)) \\ &= x[p_1(x) + p_2(x)] \\ &= x p_1(x) + x p_2(x) \\ T(p_1 + p_2) &= T(p_1) + T(p_2) \end{aligned}$$

Example ⑥ A Linear Transformation Using an Inner Product

Let V be an inner product space & $\mathbf{v}_0$ be any fixed vector in V, then the transformation $T: V \to R$ defined by $T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v}_0 \rangle$ (that maps a vector $\mathbf{x}$ into its inner product with $\mathbf{v}_0$) is a linear transformation.

Because if k is any scalar, and if $\mathbf{u}$ & $\mathbf{v}$ are any vectors in V, then it follows from the properties of inner products that

$$\begin{aligned} T(k\mathbf{u}) &= \langle k\mathbf{u}, \mathbf{v}_0 \rangle \\ &= k\langle \mathbf{u}, \mathbf{v}_0 \rangle \\ T(k\mathbf{u}) &= k T(\mathbf{u}) \end{aligned}$$

and

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= \langle \mathbf{u}+\mathbf{v}, \mathbf{v}_0 \rangle \\ &= \langle \mathbf{u}, \mathbf{v}_0 \rangle + \langle \mathbf{v}, \mathbf{v}_0 \rangle \\ T(\mathbf{u}+\mathbf{v}) &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

Example 7 **Transformations on Matrix Spaces** :- Let M_{nn} be the vector space of all $n \times n$ matrices. In each part, determine whether the transformation is linear →

(i) $T_1(A) = A^T$ (ii) $T_2(A) = \det(A)$.

Soln. (i) It follows that

$$T_1(kA) = (kA)^T = kA^T$$

i.e.; $T_1(kA) = kT_1(A)$

and $T_1(A+B) = (A+B)^T = A^T + B^T$

i.e.; $T_1(A+B) = T_1(A) + T_1(B)$

Thus, T_1 is Linear.

(ii) It follows that $T_2(kA) = \det(kA)$

$$= k^n \det(A)$$

$$= k^n T_2(A)$$

Thus, T_2 is not homogeneous and hence not Linear if $n > 1$.

Note that Additivity also fails because $\det(A+B)$ and $\det(A) + \det(B)$ are not generally equal.

Example 8 **Translation is Not Linear**

Part (i) of Theorem 1 states that a linear transformation maps $\mathbf{0}$ to $\mathbf{0}$. This property is useful for identifying transformations that are not Linear.

For example, if $\mathbf{x}_0$ is a fixed non-zero vector in R^2, then the transformation

$$T(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$$

has the geometric effect of translating each point $\mathbf{x}$ in a direction parallel to $\mathbf{x}_0$ through a distance of $\|\mathbf{x}_0\|$.

This transformation cannot be Linear since $T(\mathbf{0}) = \mathbf{x}_0$, so T does not map $\mathbf{0}$ to $\mathbf{0}$.