In Sec (5.2), we found conditions that guaranteed the diagonalizability of an $n \times n$ matrix but we did not consider what class or classes of matrices might actually satisfy those conditions. In this chapter, we will show that every Symmetric matrix is Diagonalizable. This is an extremely important result because many applications utilize it in some essential way.

SEC (7.1) ORTHOGONAL MATRICES.

In this Section, we will discuss the class of matrices whose inverses can be obtained by transposition. Such matrices occur in a variety of applications and arise as well as transition matrices when one orthonormal basis is changed to another.

Definition: A square matrix 'A' is said to be Orthogonal if its transpose is the same as its inverse, that is, if $A^{-1} = A^T$

or, equivalently, if $AA^T = A^TA = I$ —— (1)

Example (1) A 3x3 Orthogonal Matrix

The matrix

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

is Orthogonal, since

$$A^TA = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Example (2) Rotation and Reflection Matrices are Orthogonal

Recall from Sec (4.9) that the standard matrix for the counterclockwise rotation of R^2 through an angle θ is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This matrix is Orthogonal for all choices of θ since

$$A^TA = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can verify that Reflection matrices and Rotation matrices given in Tables of Sec (4.9) are all Orthogonal.

NOTE: Observe that for Orthogonal matrices in Example (1) & (2), both the row vectors and column vectors form orthonormal sets with respect to Euclidean inner product. This is a consequence of the following theorem —

THEOREM ① The following statements are equivalent for an $n \times n$ matrix 'A' —

(i) 'A' is Orthogonal.

(ii) The row vectors of 'A' form an orthonormal set in R^n with respect to Euclidean inner product.

(iii) The column vectors of 'A' form an orthonormal set in R^n with respect to Euclidean inner product.

PROPERTIES OF ORTHOGONAL MATRICES :- The following theorem lists three more fundamental properties of orthogonal matrices —

THEOREM ② (i) The Inverse of an orthogonal matrix is orthogonal.

(ii) The product of orthogonal matrices is orthogonal.

(iii) If matrix A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$.

Example ③ $\det(A) = \pm 1$ FOR AN ORTHOGONAL MATRIX 'A'

The matrix $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is orthogonal since its row (and column) vectors form orthonormal sets in R^2 with the Euclidean inner product.

$$\det(A) = \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{2}}\right) - \left(-\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{2}}\right)$$

$$= \tfrac{1}{2} + \tfrac{1}{2} = 1$$

Inchanging the rows of A produces an orthogonal matrix whose determinant is -1.

ORTHOGONAL MATRICES AS LINEAR OPERATORS - We observed in Example ② that standard matrices for the basic reflection and rotation operators on R^2 and R^3 are Orthogonal. The next theorem will explain why this is so.

THEOREM ③ If 'A' is an $n \times n$ matrix, then the following statements are equivalent —

(i) 'A' is Orthogonal.

(ii) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n.

(iii) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ in R^n.

Theorem ③ has a useful geometric interpretation when considered from the viewpoint of matrix transformations. If 'A' is an orthogonal matrix and $T_A : R^n \to R^n$ is multiplication by A, then we will call T_A an Orthogonal operator on R^n. It follows from parts (i) & (ii) of Theorem ③ that the orthogonal operators on R^n are precisely those operators that leave the lengths of all vectors unchanged. This explains why, in Example ②, we found the standard matrices for the basic reflections and rotations of R^2 and R^3 to be orthogonal.

Parts (i) & (iii) of Theorem ③ imply that orthogonal operators leave the angle between two vectors unchanged.

SEC (7.2) ORTHOGONAL DIAGONALIZATION

In this Section, we will be concerned with the problem of diagonalizing a symmetric matrix A. As we will see, this problem is closely related to that of finding an orthonormal basis for R^n that consists of eigenvectors of A. Problems of this type are important because many of the matrices that arise in applications are symmetric.

THE ORTHOGONAL DIAGONALIZATION PROBLEM

In a Definition of Sec (5.2) we defined two square matrices, A and B to be Similar if there is an invertible matrix P such that $P^{-1}AP = B$. In this Section, we will be concerned with the special case in which it is possible to find an orthogonal matrix P for which this relationship holds.

We begin with the following definition —

Definition: If A & B are square matrices, then we say that A & B are Orthogonally Similar if there is an orthogonal matrix P such that $P^TAP = B$.

If the matrix A is orthogonally similar to some diagonal matrix, say $P^TAP = D$ then we say that 'A' is orthogonally diagonalizable and that 'P' orthogonally diagonalizes 'A'.

Our first goal in this Section is to determine what conditions a matrix must satisfy to be orthogonally diagonalizable. As a first step, observe that there is no hope of orthogonally diagonalizing a matrix that is not symmetric.

CONDITIONS FOR ORTHOGONAL DIAGONALIZABILITY

The following theorem shows that every symmetric matrix is, in fact, orthogonally diagonalizable. In this theorem and for the remainder of this Section, orthogonal will mean orthogonal with respect to the Euclidean inner product on R^n.

THEOREM (1) If A is an $n \times n$ matrix, then the following statements are equivalent —

(i) 'A' is orthogonally diagonalizable.

(ii) 'A' has an orthonormal set of n eigenvectors.

(iii) 'A' is symmetric.

PROPERTIES OF SYMMETRIC MATRICES

Our next goal is to devise a procedure for orthogonally diagonalizing a symmetric matrix, but before we can do so, we need the following critical theorem about eigenvalues and eigenvectors of the symmetric matrices.

THEOREM (2) If A is a symmetric matrix, then

(i) The eigenvalues of A are all real numbers.

(ii) Eigenvectors from different eigenspaces are Orthogonal.

ORTHOGONALLY DIAGONALIZING AN $n \times n$ SYMMETRIC MATRICES.

Step① Find a basis for each eigenspace of matrix A.

Step② Apply Gram-Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.

Step③ Form the matrix P whose columns are the vectors constructed in Step②. This matrix will orthogonally diagonalize matrix A and the eigenvalues on the diagonal of $D = P^TAP$ will be in the same order as their corresponding eigenvectors in P.

REMARK: The justification of this procedure should be clear :- Theorem ② ensures that eigenvectors from different eigenspaces are orthogonal and applying the Gram-Schmidt process ensures that the eigenvectors within the same eigenspace are Orthonormal. It follows that the entire set of eigenvectors obtained by this procedure will be Orthonormal.