In Chapter (3), we defined the dot product of vectors in R^n and we used that concept to define notions of length, angle, distance and orthogonality.

In this Chapter, we will generalize those ideas so they are applicable in any vector space, not just R^n.

SEC (6.1) INNER PRODUCTS

In this Section, we will use the most important properties of the dot product on R^n as axioms, which if satisfied by the vectors in a vector space V, will enable us to extend the notions of length, distance, angle and perpendicularity to general vector spaces.

Definition: An Inner Product on a real vector space V is a function that associates a real number $\langle u, v \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors u, v and w in V and all scalars k

(i) $\langle u, v \rangle = \langle v, u \rangle$ [Symmetry Axiom]

(ii) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ [Additivity Axiom]

(iii) $\langle ku, v \rangle = k\langle u, v \rangle$ [Homogeneity Axiom]

(iv) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $\mathbf{v} = \mathbf{0}$ [Positivity Axiom]

A real vector space with an inner product is called a Real Product Space.

NOTE: This defi. applies only to real vector spaces. Since we will have little need for complex vector spaces from this point on, you can assume that all vector spaces under discussion are real, even though some of theorems are also valid in complex vector spaces.

Because the axioms for a real inner product space are based on properties of the dot product, these inner product space axioms will be satisfied automatically if we define the inner product of two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n to be | $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ & $\mathbf{v} = (v_1, v_2, \cdots, v_n)$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

This inner product is commonly called the Euclidean Inner Product (or Standard Inner Product) on R^n to distinguish it from other possible inner products that might be defined on R^n. We call R^n with Euclidean inner product as Euclidean n-space.

Inner products can be used to define notions of norm and distance in a general inner product space just as we did with dot products in R^n. Recall from Sec (3.2) that if u and v are vectors in Euclidean n-space, then norm and distance can be expressed in terms of the dot product as

$$\|v\| = \sqrt{v \cdot v}$$

and $$d(u, v) = \|u - v\| = \sqrt{(u-v) \cdot (u-v)}$$

Motivated by these formulas, we make the following definition —

Definition: If V is a real inner product space, then the Norm (or Length) of a vector $\mathbf{v}$ in V is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

and the Distance between two vectors is denoted by $d(\mathbf{u}, \mathbf{v})$ and is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{\langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v} \rangle}$$

A vector of norm 1 is called a Unit Vector.

The following Theorem shows that norms and distances in real inner product spaces have many of the properties that you might expect —

THEOREM (1) If $\mathbf{u}$ & $\mathbf{v}$ are vectors in a real inner product space V, and if k is a scalar, then

(i) $\|\mathbf{v}\| \geq 0$ with equality iff $\mathbf{v} = 0$.

(ii) $\|k\mathbf{v}\| = |k| \, \|\mathbf{v}\|$.

(iii) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$.

(iv) $d(\mathbf{u}, \mathbf{v}) \geq 0$ with equality iff $\mathbf{u} = \mathbf{v}$.

WEIGHTED EUCLIDEAN INNER PRODUCT:

Although the Euclidean inner product is the most important inner product on R^n, there are various applications in which it is desirable to modify it by weighting each term differently.

More precisely, if $w_1, w_2, \cdots, w_n$ are positive real numbers, which we will call weights, and if $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ are vectors in R^n, then it can be shown that the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n \qquad \text{—— (1)}$$

defines an inner product on R^n that we call Weighted Euclidean Inner Product with the weights $w_1, w_2, \cdots, w_n$.

Note that the standard Euclidean inner product is the special case of the weighted Euclidean inner product in which all the weights are 1.

Example (1) **Weighted Euclidean Inner Product**

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in R^2. Verify that the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ satisfies the four inner product axioms.

Soln. Axiom (1) We have

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2 = 3v_1u_1 + 2v_2u_2$$

i.e., $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

Axiom (2) If $\mathbf{w} = (w_1, w_2)$ then

$$\begin{aligned} \langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle &= 3(u_1+v_1)w_1 + 2(u_2+v_2)w_2 \\ &= 3(u_1w_1 + v_1w_1) + 2(u_2w_2 + v_2w_2) \\ &= (3u_1w_1 + 2u_2w_2) + (3v_1w_1 + 2v_2w_2) \\ &= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle \end{aligned}$$

Axiom (3) We have $\langle k\mathbf{u},\mathbf{v}\rangle = 3(ku_1)v_1 + 2(ku_2)v_2$

$$= k(3u_1v_1 + 2u_2v_2)$$

i.e., $\langle k\mathbf{u},\mathbf{v}\rangle = k\langle \mathbf{u},\mathbf{v}\rangle$

Axiom (4) $\langle \mathbf{v},\mathbf{v}\rangle = 3(v_1v_1) + 2(v_2v_2)$

$= 3v_1^2 + 2v_2^2 \geq 0$ with equality iff $v_1 = v_2 = 0$, i.e., iff $\mathbf{v} = \mathbf{0}$.

NOTE: It is important to keep in mind that Norm and Distance depend on the inner product being used. If the inner product is changed, then the Norms and Distances between vectors also change.

For example, for the vectors $\mathbf{u} = (1,0)$ and $\mathbf{v} = (0,1)$ in R^2 with Euclidean inner product $\langle \mathbf{u},\mathbf{v}\rangle = u_1v_1 + u_2v_2$

we have $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u},\mathbf{u}\rangle} = \sqrt{(1)(1)+(0)(0)} = 1$

and $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|(1,-1)\|$

$$= \sqrt{\langle (1,-1),(1,-1)\rangle}$$

$$= \sqrt{(1)(1)+(-1)(-1)} = \sqrt{2}$$

but if we change to the weighted Euclidean inner product

$$\langle \mathbf{u},\mathbf{v}\rangle = 3u_1v_1 + 2u_2v_2$$

then we have $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u},\mathbf{u}\rangle}$

$$= \sqrt{\langle (1,0),(1,0)\rangle}$$

$$= \sqrt{3(1)(1) + 2(0)(0)} = \sqrt{3}$$

and $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$

$$= \|(1,-1)\|$$

$$= \sqrt{\langle (1,-1),(1,-1)\rangle}$$

$$= \sqrt{3(1)(1) + 2(-1)(-1)}$$

$$= \sqrt{5}$$

UNIT CIRCLES AND SPHERES IN INNER PRODUCT SPACES.

If V is an inner product space, then the set of points in V that satisfy $||\mathbf{u}|| = 1$ is called the Unit Sphere or sometimes the Unit Circle in V.

Example 3 Unusual Unit Circles in R^2

(i) Sketch the unit circle in an xy-coordinate system in R^2 using Euclidean inner product

$$<\mathbf{u},\mathbf{v}> = u_1v_1 + u_2v_2 .$$

(ii) Sketch the unit circle in an xy-coordinate system in R^2 using weighted Euclidean inner product $\quad <\mathbf{u},\mathbf{v}> = \frac{1}{9}u_1v_1 + \frac{1}{4}u_2v_2 .$

Soln. (i) If $\mathbf{u} = (x, y)$ in R^2, then

$$||\mathbf{u}|| = \sqrt{<\mathbf{u},\mathbf{u}>}$$

$$= \sqrt{<(x,y),(x,y)>}$$

$$= \sqrt{(x)(x)+(y)(y)} = \sqrt{x^2+y^2}$$

so the equ. of unit circle is $||\mathbf{u}|| = 1$

i.e., $\sqrt{x^2+y^2} = 1$

$\Rightarrow \quad x^2+y^2 = 1$

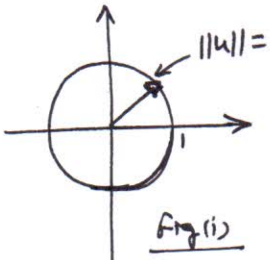

Fig (i)

The Unit Circle using Standard Euclidean Inne

As expected, the graph of this equ. is a circle of radius 1 centered at Origin (See Fig (i))

Soln (ii) If $\mathbf{u} = (x, y) \in R^2$, then

$$||\mathbf{u}|| = \sqrt{<\mathbf{u},\mathbf{u}>}$$

$$= \sqrt{<(x,y),(x,y)>}$$

$$= \sqrt{\tfrac{1}{9}(x)(x) + \tfrac{1}{4}(y)(y)}$$

$$= \sqrt{\tfrac{1}{9}x^2 + \tfrac{1}{4}y^2}$$

so the equ. of unit circle is $||\mathbf{u}|| = 1$

i.e., $\sqrt{\frac{1}{9}x^2 + \frac{1}{4}y^2} = 1$

$\Rightarrow \quad \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

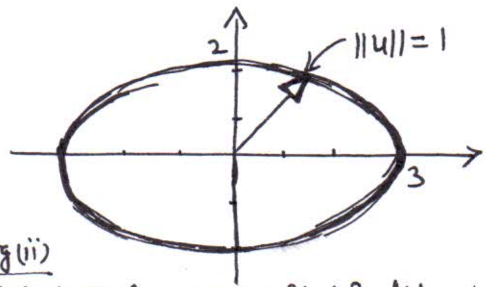

Fig (ii)

The Unit Circle using weighted Euclidean inner produ

The graph of this equ. is the ellipse shown in Fig (ii).