SEC (5.1) EIGENVALUES AND EIGENVECTORS.

In this Section, we will define the notions of 'Eigenvalue' and 'Eigenvector' and discuss some of their basic properties.

Definition of Eigenvalue and Eigenvector

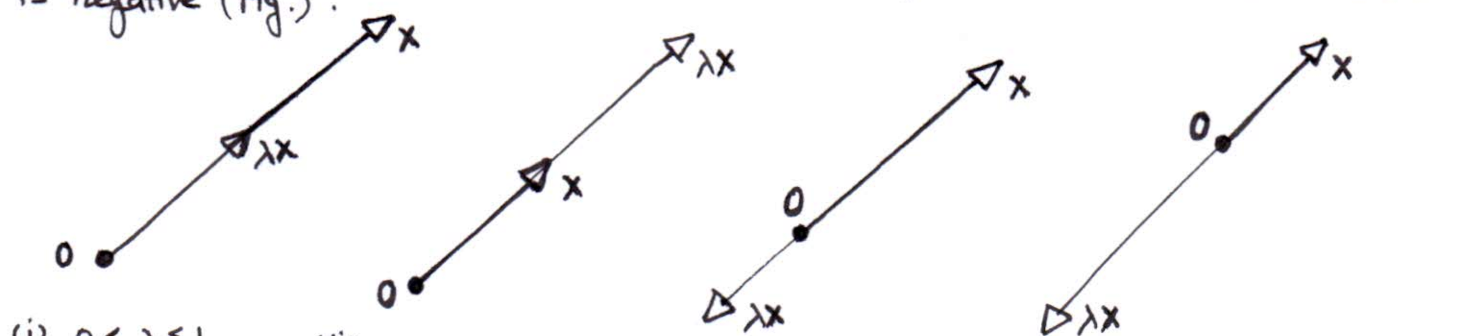
If 'A' is an $n \times n$ matrix, then a non-zero vector $\mathbf{x}$ in R^n is called an Eigenvector of 'A' (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$, that is,

$$A\mathbf{x} = \lambda \mathbf{x} \text{ , for some scalar } \lambda .$$

The scalar λ is called an Eigenvalue of 'A' (or of T_A), and $\mathbf{x}$ is said to be an Eigenvector corresponding to λ.

NOTE: In general, the image of a vector $\mathbf{x}$ under multiplication by a square matrix 'A' differs from $\mathbf{x}$ in both magnitude and direction. However, in the special case where $\mathbf{x}$ is an eigenvector of 'A', multiplication by 'A' leaves the direction unchanged.

For example, in R^2 or R^3, multiplication by 'A' maps each eigenvector $\mathbf{x}$ of A (if any) along the same line through the origin as $\mathbf{x}$. Depending on the sign and magnitude of the eigenvalue λ corresponding to $\mathbf{x}$, the operation $A\mathbf{x} = \lambda\mathbf{x}$ compresses or stretches $\mathbf{x}$ by a factor of λ, with a reversal of direction in the case where λ is negative (fig.).



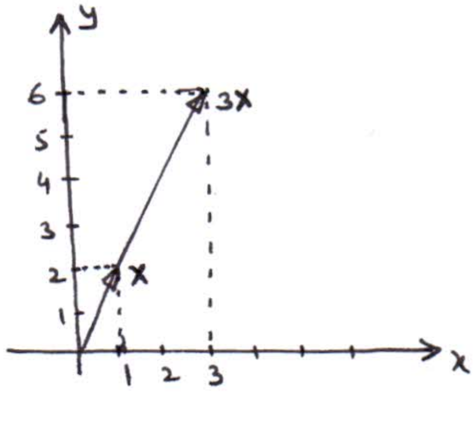
(i) $0 \le \lambda \le 1$ (ii) $\lambda \ge 1$ (iii) $-1 \le \lambda \le 0$ (iv) $\lambda \le -1$

Example ① Eigenvector of a 2x2 Matrix

The vector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an Eigenvector of matrix $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ corresp. to eigenvalue $\lambda = 3$

since
$$\begin{aligned} A\mathbf{x} &= \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\ &= 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= 3\mathbf{x} \end{aligned}$$

Geometrically, multiplication by A has stretched the vector $\mathbf{x}$ by a factor of 3 (See fig.)



COMPUTING EIGENVALUES AND EIGENVECTORS

Our next objective is to obtain a general procedure for finding Eigenvalues and Eigenvectors of an $n\times n$ matrix 'A'. We will begin with the problem of finding the Eigenvalues of 'A'.

Note first that the equ. $A\mathbf{x} = \lambda \mathbf{x}$ can be rewritten as $A\mathbf{x} = \lambda I \mathbf{x}$

or equivalently as $(\lambda I - A)\mathbf{x} = 0$

For λ to be an Eigenvalue of A, this equ. must have a non-zero solu. for $\mathbf{x}$. But it follows from a Theorem of Chapter ④ that this is so iff the coefficient matrix $(\lambda I - A)$ has a zero determinant. Thus, we have the following result —

THEOREM① If 'A' is an $n\times n$ matrix, then λ is an Eigenvalue of A iff it satisfies the equ.

$$\det(\lambda I - A) = 0 \quad \text{——— ①}$$

This equ. is called Characteristic Equation of matrix A.

Example ② Use Characteristic Equ. to find all eigenvalues of matrix $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$.

Solu. The eigenvalues of the matrix A are the solutions of the equ.

$$|\lambda I - A| = 0$$

i.e., $\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\right) = 0$

i.e., $\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\right) = 0$

i.e., $\begin{vmatrix} \lambda-3 & 0 \\ -8 & \lambda+1 \end{vmatrix} = 0$

$$\Rightarrow \quad (\lambda-3)(\lambda+1) = 0 \quad \Rightarrow \quad \underline{\lambda = 3, -1}$$

NOTE: When $\det(\lambda I - A)$ on left side of ① is expanded, the result is a polynomial $p(\lambda)$ of degree n, that is called Characteristic Polynomial of matrix A.

For example, it follows that the characteristic polynomial of 2×2 matrix A in Example ② is

$$p(\lambda) = (\lambda-3)(\lambda+1)$$
$$= \lambda^2 - 2\lambda - 3, \text{ which is a polynomial of degree 2.}$$

In general, the Characteristic Polynomial of an $n\times n$ matrix has the form

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n, \text{ in which the coeff. of } \lambda^n \text{ is } 1.$$

Since a polynomial of degree n has at most n distinct roots, it follows that the equ.

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0 \quad \text{——— ②}$$

has at most n distinct solutions and consequently that an $n\times n$ matrix has at most n distinct eigenvalues. Since some of these solu. may be complex numbers, it is possible for a matrix to have complex eigenvalues, even if that matrix itself has real entries.

Example ③ Eigenvalues of a 3x3 Matrix

Find the Eigenvalues of the matrix, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$.

Soln. The Characteristic Polynomial of matrix A is

$$\det(\lambda I - A)$$

$$= \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} \right)$$

$$= \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{vmatrix}$$

$$= \lambda[\lambda(\lambda-8)+17] - (-1)[0-(-4)(-1)]$$

$$= \lambda(\lambda^2 - 8\lambda + 17) + (0-4)$$

$$= \lambda^3 - 8\lambda^2 + 17\lambda - 4$$

The Eigenvalues of A must therefore satisfy the cubic eqn.

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0 \quad \text{———} \quad ①$$

putting $\lambda = 4$ in L.H.S. of ①, we get

> We try only divisors of const. term (i.e. -4) which are $\pm 1, \pm 2, \pm 4$

$$\text{L.H.S.} = (4)^3 - 8(4)^2 + 17(4) - 4$$

$$= 64 - 128 + 68 - 4 = 0 = \text{R.H.S.}$$

so $\lambda - 4$ must be a factor of L.H.S. of ①

Now dividing $\lambda^3 - 8\lambda^2 + 17\lambda - 4$ by $\lambda - 4$ as:-

$$\begin{array}{r|l} & \lambda^2 - 4\lambda + 1 \\ \hline \lambda - 4 & \lambda^3 - 8\lambda^2 + 17\lambda - 4 \\ & \underline{\lambda^3 - 4\lambda^2} \\ & \quad -4\lambda^2 + 17\lambda - 4 \\ & \quad \underline{-4\lambda^2 + 16\lambda} \\ & \qquad \lambda - 4 \\ & \qquad \underline{\lambda - 4} \\ & \qquad 0 \end{array}$$

Hence ① can be rewritten as

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

$$\Rightarrow \lambda = 4 \quad \text{and} \quad \lambda = \frac{4 \pm \sqrt{(-4)^2 - 4}}{2}$$

i.e., $\lambda = 4$ and $\lambda = \frac{4 \pm \sqrt{12}}{2}$

i.e., $\lambda = 4$ and $\lambda = \frac{4 \pm 2\sqrt{3}}{2}$

i.e., $\lambda = 4$ and $\lambda = 2 \pm \sqrt{3}$

Thus, Eigenvalues of matrix A are $\lambda_1 = 4$, $\lambda_2 = 2 + \sqrt{3}$ and $\lambda_3 = 2 - \sqrt{3}$.

Example ④ Eigenvalues of an Upper Triangular Matrix

Find the Eigenvalues of the upper triangular matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$

Soln. Recalling that the determinant of a triangular matrix is equal to the product of the entries on the main diagonal, we obtain

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}\right)$$

$$= \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ 0 & \lambda - a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & \lambda - a_{33} & -a_{34} \\ 0 & 0 & 0 & \lambda - a_{44} \end{vmatrix}$$

$$= (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44})$$

Thus, the Characteristic Eqn. is

$$(\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}) = 0$$

$$\Rightarrow \lambda = a_{11},\ \lambda = a_{22},\ \lambda = a_{33},\ \lambda = a_{44}$$

Thus, the Eigenvalues are $\lambda_1 = a_{11},\ \lambda_2 = a_{22},\ \lambda_3 = a_{33},\ \lambda_4 = a_{44}$

which are precisely the diagonal entries of matrix A.

The following general theorem should be evident from the computations in preceding example –

THEOREM ②: If 'A' is $n \times n$ triangular matrix (upper triangular, lower triangular or diagonal), then the Eigenvalues of matrix A are the entries on the main diagonal of 'A'.

Example ⑤ Eigenvalues of a Lower Triangular Matrix

Find the eigenvalues of lower triangular matrix $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 2/3 & 0 \\ 5 & -8 & -1/4 \end{bmatrix}$

Soln. By Theorem ②, the eigenvalues of matrix A are

$$\lambda_1 = \frac{1}{2},\quad \lambda_2 = \frac{2}{3},\quad \lambda_3 = -\frac{1}{4}.$$

THEOREM ③ If A is an $n \times n$ matrix, the following statements are equivalent –

(i) 'λ' is an eigenvalue of matrix A.

(ii) The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has non-trivial solutions.

(iii) There is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda \mathbf{x}$.

(iv) 'λ' is a solution of the characteristic eqn. $\det(\lambda I - A) = 0$.