SEC (4.1) REAL VECTOR SPACES

In this Section, we will extend the concept of a vector by using the basic properties of vectors in R^n as axioms, which if satisfied by a set of objects, guarantee that those objects behave like familiar vectors.

VECTOR SPACE AXIOMS -- The following definition consists of ten axioms, eight of which are properties of vectors in R^n that were stated in theorem of Sec (3.1).

Definition : Let V be an arbitrary non-empty set of objects on which two operations are defined -- Addition, and Multiplication by scalars.

By Addition, we mean a rule for associating with each pair of objects $\mathbf{u}$ & $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the Sum of $\mathbf{u}$ & $\mathbf{v}$;

By Scalar Multiplication, we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called Scalar Multiple of $\mathbf{u}$ by k.

If the following axioms are satisfied by all objects $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars k and m, then we call V a vector space and we call objects in V as vectors.

(1) If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.

(2) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(3) $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$

(4) There is an object $\mathbf{0}$ in V, called zero vector for V, such that

$$\mathbf{0}+\mathbf{u} = \mathbf{u}+\mathbf{0} = \mathbf{u} \text{ for all } \mathbf{u} \text{ in } V$$

(5) For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called negative of $\mathbf{u}$, such that

$$\mathbf{u}+(-\mathbf{u}) = (-\mathbf{u})+\mathbf{u} = \mathbf{0}.$$

(6) If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.

(7) $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$

(8) $(k+m)\mathbf{u} = k\mathbf{u}+m\mathbf{u}$

(9) $k(m\mathbf{u}) = (km)\mathbf{u}$

(10) $1.\mathbf{u} = \mathbf{u}$

Our first example is the simplest of all vector spaces in that it contains only one object. Since Axiom ④ requires that every vector space contain a zero vector, the object will have to be that vector.

Example ① (The Zero Vector Space)

Let V consist of a single object, which we denote by $\mathbf{0}$ and define

$$\mathbf{0}+\mathbf{0}=\mathbf{0}$$

and $\quad k\mathbf{0}=\mathbf{0}$ for all scalars k.

It is easy to check that all vector axioms are satisfied. We call this the Zero Vector Space.

Example ② (R^n is a Vector space)

Let $V=R^n$, and define the vector space operations on V to be the usual operations of addition and scalar multiplication of n-tuples; that is,

$$\begin{aligned}\mathbf{u}+\mathbf{v} &= (u_1,u_2,\dots,u_n)+(v_1,v_2,\dots,v_n)\\ &= (u_1+v_1,u_2+v_2,\dots,u_n+v_n)\end{aligned}$$

$$\& \quad k\mathbf{u}=(ku_1,ku_2,\dots,ku_n)$$

The set $V=R^n$ is closed under Addition and Scalar multiplication because the foregoing operations produce n-tuples as their end result.

Axiom ②

$$\begin{aligned}\mathbf{u}+\mathbf{v} &= (u_1,u_2,\dots,u_n)+(v_1,v_2,\dots,v_n)\\ &= (u_1+v_1,u_2+v_2,\dots,u_n+v_n)\\ &= (v_1+u_1,v_2+u_2,\dots,v_n+u_n)\\ &= (v_1,v_2,\dots,v_n)+(u_1,u_2,\dots,u_n)\\ &= \mathbf{v}+\mathbf{u}\end{aligned}$$

Axiom ③

$$\begin{aligned}\mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (u_1,u_2,\dots,u_n)+\left[(v_1,v_2,\dots,v_n)+(w_1,w_2,\dots,w_n)\right]\\ &= (u_1,u_2,\dots,u_n)+(v_1+w_1,v_2+w_2,\dots,v_n+w_n)\\ &= \left[u_1+(v_1+w_1),u_2+(v_2+w_2),\dots,u_n+(v_n+w_n)\right]\\ &= \left[(u_1+v_1)+w_1,(u_2+v_2)+w_2,\dots,(u_n+v_n)+w_n\right]\\ &= (u_1+v_1,u_2+v_2,\dots,u_n+v_n)+(w_1,w_2,\dots,w_n)\\ &= \left[(u_1,u_2,\dots,u_n)+(v_1,v_2,\dots,v_n)\right]+(w_1,w_2,\dots,w_n)\\ &= (\mathbf{u}+\mathbf{v})+\mathbf{w}\end{aligned}$$

Axiom ④ $\quad \forall\, \mathbf{u}=(u_1,u_2,\dots,u_n)\in V$, we have

$$\begin{aligned}\mathbf{u}+\mathbf{0} &= (u_1,u_2,\dots,u_n)+(0,0,\dots,0)\\ &= (u_1+0,u_2+0,\dots,u_n+0)\\ &= (u_1,u_2,\dots,u_n)\\ &= \mathbf{u}\end{aligned}$$

Axiom ⑤ $\forall \mathbf{u} = (u_1, u_2, \dots, u_n) \in V$, we have $-\mathbf{u} = (-u_1, -u_2, \dots, -u_n)$ such that

$$\begin{aligned}
\mathbf{u} + (-\mathbf{u}) &= (u_1, u_2, \dots, u_n) + (-u_1, -u_2, \dots, -u_n) \\
&= (u_1 - u_1,\ u_2 - u_2,\ \dots,\ u_n - u_n) \\
&= (0, 0, \dots, 0) = \mathbf{0}
\end{aligned}$$

Axiom ⑦

$$\begin{aligned}
k(\mathbf{u} + \mathbf{v}) &= k\left[(u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n)\right] \\
&= k(u_1 + v_1,\ u_2 + v_2,\ \dots,\ u_n + v_n) \\
&= \left[k(u_1 + v_1),\ k(u_2 + v_2),\ \dots,\ k(u_n + v_n)\right] \\
&= (ku_1 + kv_1,\ ku_2 + kv_2,\ \dots,\ ku_n + kv_n) \\
&= (ku_1, ku_2, \dots, ku_n) + (kv_1, kv_2, \dots, kv_n) \\
&= k(u_1, u_2, \dots, u_n) + k(v_1, v_2, \dots, v_n) \\
&= k\mathbf{u} + k\mathbf{v}
\end{aligned}$$

Axiom ⑧

$$\begin{aligned}
(k+m)\mathbf{u} &= (k+m)(u_1, u_2, \dots, u_n) \\
&= \left((k+m)u_1,\ (k+m)u_2,\ \dots,\ (k+m)u_n\right) \\
&= (ku_1 + mu_1,\ ku_2 + mu_2,\ \dots,\ ku_n + mu_n) \\
&= (ku_1, ku_2, \dots, ku_n) + (mu_1, mu_2, \dots, mu_n) \\
&= k(u_1, u_2, \dots, u_n) + m(u_1, u_2, \dots, u_n) \\
&= k\mathbf{u} + m\mathbf{u}
\end{aligned}$$

Axiom ⑨

$$\begin{aligned}
k(m\mathbf{u}) &= k\left[m(u_1, u_2, \dots, u_n)\right] \\
&= k(mu_1, mu_2, \dots, mu_n) \\
&= (kmu_1,\ kmu_2,\ \dots,\ kmu_n) \\
&= km(u_1, u_2, \dots, u_n) \\
&= (km)\mathbf{u}
\end{aligned}$$

Axiom ⑩

$$\begin{aligned}
1.\mathbf{u} &= 1.(u_1, u_2, \dots, u_n) \\
&= (1.u_1,\ 1.u_2,\ \dots,\ 1.u_n) \\
&= (u_1, u_2, \dots, u_n) \\
&= \mathbf{u}.
\end{aligned}$$

Example ③ (The Vector Space of Infinite Sequences of Real Numbers) ie: R^∞

Let V consists of objects of the form $\mathbf{u} = (u_1, u_2, ---, u_n, ---)$ in which $u_1, u_2, ---, u_n, ---$ is an infinite sequence of real numbers.

We define Addition and Scalar multiplication componentwise by

$$\mathbf{u}+\mathbf{v} = (u_1, u_2, ---, u_n, ---) + (v_1, v_2, ---, v_n, ---)$$
$$= (u_1+v_1, u_2+v_2, ----, u_n+v_n, ----)$$

& $$k\mathbf{u} = (ku_1, ku_2, ----, ku_n, ---)$$

This is left as an exercise to confirm that V with these operations is a Vector Space.
We will denote this vector space by the symbol R^∞.

Example ④ (A Vector Space of 2x2 Matrices)

Let V is the set of 2x2 matrices with real entries, and take the vector space operations on V to be the usual operations of matrix addition and scalar multiplication; that is,

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$

& $$k\mathbf{u} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

The set V is closed under Addition and Scalar multiplication because the foregoing operations produce 2x2 matrices as the end result.

Thus, it remains to confirm that Axioms 2, 3, 4, 5, 7, 8, 9 and 10 hold.

Axiom ②

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$
$$= \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$
$$= \begin{bmatrix} v_{11}+u_{11} & v_{12}+u_{12} \\ v_{21}+u_{21} & v_{22}+u_{22} \end{bmatrix}$$
$$= \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{v}+\mathbf{u}$$

Axiom ③

$$\mathbf{u}+(\mathbf{v}+\mathbf{w}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11}+w_{11} & v_{12}+w_{12} \\ v_{21}+w_{21} & v_{22}+w_{22} \end{bmatrix}$$
$$= \begin{bmatrix} u_{11}+(v_{11}+w_{11}) & u_{12}+(v_{12}+w_{12}) \\ u_{21}+(v_{21}+w_{21}) & u_{22}+(v_{22}+w_{22}) \end{bmatrix}$$
$$= \begin{bmatrix} (u_{11}+v_{11})+w_{11} & (u_{12}+v_{12})+w_{12} \\ (u_{21}+v_{21})+w_{21} & (u_{22}+v_{22})+w_{22} \end{bmatrix}$$
$$= \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$
$$= (\mathbf{u}+\mathbf{v})+\mathbf{w}$$

Axiom 4 for each $\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ in V, we have $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in V

such that $\mathbf{0} + \mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$

and similarly $\mathbf{u} + \mathbf{0} = \mathbf{u}$

Axiom 5 for each object $\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ in V, we have $-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$ in V

such that

$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11}-u_{11} & u_{12}-u_{12} \\ u_{21}-u_{21} & u_{22}-u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

and similarly $(-\mathbf{u}) + \mathbf{u} = \mathbf{0}$

Axiom 7

$$k(\mathbf{u}+\mathbf{v}) = k \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ku_{11}+kv_{11} & ku_{12}+kv_{12} \\ ku_{21}+kv_{21} & ku_{22}+kv_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} + \begin{bmatrix} kv_{11} & kv_{12} \\ kv_{21} & kv_{22} \end{bmatrix}$$

$$= k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + k \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

$$= k\mathbf{u} + k\mathbf{v}$$

Axiom 8

$$(k+m)\mathbf{u} = (k+m) \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

$$= \begin{bmatrix} (k+m)u_{11} & (k+m)u_{12} \\ (k+m)u_{21} & (k+m)u_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ku_{11}+mu_{11} & ku_{12}+mu_{12} \\ ku_{21}+mu_{21} & ku_{22}+mu_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} + \begin{bmatrix} mu_{11} & mu_{12} \\ mu_{21} & mu_{22} \end{bmatrix}$$

$$= k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + m \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = k\mathbf{u} + m\mathbf{u}$$

Axiom 9

$$k(m\mathbf{u}) = k \begin{bmatrix} mu_{11} & mu_{12} \\ mu_{21} & mu_{22} \end{bmatrix} = (km) \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = (km)\mathbf{u}$$

Axiom 10

$$1.\mathbf{u} = 1 \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} 1.u_{11} & 1.u_{12} \\ 1.u_{21} & 1.u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$