Example: Find the co-ordinate vector of $\mathbf{W} = (1,1)$ relative to the basis $\{\mathbf{u}_1 = (1,-1),\ \mathbf{u}_2 = (1,1)\}$ for R^2.

Solution: To find $(\mathbf{W})_S$, we must first express $\mathbf{W}$ as a linear combination of vectors in basis ie;, we must find values c_1 & c_2 such that

$$\mathbf{W} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2$$

ie;, $$(1,1) = c_1(1,-1) + c_2(1,1)$$

$$\Rightarrow \quad (1,1) = (c_1 + c_2,\ -c_1 + c_2)$$

Equating the components on both sides

$$c_1 + c_2 = 1 \quad \text{——— (i)}$$
$$-c_1 + c_2 = 1 \quad \text{——— (ii)}$$

Solving (i) & (ii), we get

$$c_1 = 0, \quad c_2 = 1$$

Thus, the coordinate vector of $\mathbf{W}$ relative to basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ is $(0,1)$.

SEC (4.7) ROW SPACE, COLUMN SPACE AND NULL SPACE

In this Section, we will study some important vector spaces that are associated with matrices.

Definition: for an $m\times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the vectors $r_1 = [a_{11} \ a_{12} \cdots a_{1n}]$, $r_2 = [a_{21} \ a_{22} \cdots a_{2n}], \ldots, r_m = [a_{m1} \ a_{m2} \cdots a_{mn}]$ in R^n that are formed from the rows of A are called Row Vectors of A.

and the vectors $c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \ c_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ \ldots, \ c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$

in R^m formed from the column of A are called Column Vectors of A.

Example ① Row and Column Vectors of a 2×3 matrix

$$\text{Let } A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}_{2\times 3}$$

The row vectors of A are $r_1 = [2 \ \ 1 \ \ 0]$ and $r_2 = [3 \ -1 \ \ 4] \in R^3$

and the column vectors of A are $c_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \ c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ c_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \in R^2$

Definition: If A is an $m\times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the Row Space of A, and the subspace of R^m spanned by the column vectors of A is called the Column Space of A.

The solution space of the homogeneous system of equations $Ax = 0$, which is a subspace of R^n, is called the Null Space of A.

In this Section and the next, we will be concerned with two general questions —

Q① What relationships exist among the solutions of a linear system $Ax = b$ and the row space, column space and null space of the coefficient matrix A?

Q② What relationships exist among row space, column space & null space of a matrix?

Starting with the first question, suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

It follows from Sec 1.3 that if $c_1, c_2, \ldots, c_n$ denote the column vectors of A, then the product $A\mathbf{x}$ can be expressed as a linear combination of these vectors with coefficients from $\mathbf{x}$; that is,

$$A\mathbf{x} = x_1 c_1 + x_2 c_2 + \cdots + x_n c_n \quad \text{——— } (1)$$

Thus, a linear system, $A\mathbf{x} = \mathbf{b}$, of m equations in n unknowns can be written as

$$x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = \mathbf{b} \quad \text{——— } (2)$$

~~from~~ from which we conclude that $A\mathbf{x} = \mathbf{b}$ is Consistent iff $\mathbf{b}$ is expressible as a linear combination of the column vectors of A. This yields the following theorem –

THEOREM 1 A system of linear equations $A\mathbf{x} = \mathbf{b}$ is Consistent if and only if $\mathbf{b}$ is in the column space of A.

Example 2 A Vector '$\mathbf{b}$' in the Column Space of 'A'

Let $A\mathbf{x} = \mathbf{b}$ be a linear system
$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that $\mathbf{b}$ is in the column space of A by expressing it as a linear combination of the column vectors of A.

Soln. The Augmented matrix, $[A|b] = \left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 1 & 2 & -3 & -9 \\ 2 & 1 & -2 & -3 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -9 \\ -1 & 3 & 2 & 1 \\ 2 & 1 & -2 & -3 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -9 \\ 0 & 5 & -1 & -8 \\ 0 & -3 & 4 & 15 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -9 \\ 0 & 1 & -1/5 & -8/5 \\ 0 & -3 & 4 & 15 \end{array}\right] \quad R_2 \to \tfrac{1}{5} R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & -9 \\ 0 & 1 & -1/5 & -8/5 \\ 0 & 0 & 17/5 & 51/5 \end{bmatrix} \quad R_3 \to R_3 + 3R_2$$

The corresponding equations are

$$\left.\begin{aligned} x_1 + 2x_2 - 3x_3 &= -9 \\ x_2 - \tfrac{1}{5}x_3 &= -\tfrac{8}{5} \\ \tfrac{17}{5}x_3 &= \tfrac{51}{5} \end{aligned}\right)$$

Solving these equations, $x_1 = 2,\ x_2 = -1,\ x_3 = 3$

It follows from this and formula ② that

$$x_1c_1 + x_2c_2 + x_3c_3 = 2\begin{bmatrix}-1\\1\\2\end{bmatrix} + (-1)\begin{bmatrix}3\\2\\1\end{bmatrix} + 3\begin{bmatrix}2\\-3\\-2\end{bmatrix}$$

$$= \begin{bmatrix}-2\\2\\4\end{bmatrix} + \begin{bmatrix}-3\\-2\\-1\end{bmatrix} + \begin{bmatrix}6\\-9\\-6\end{bmatrix}$$

$$= \begin{bmatrix}-2-3+6\\2-2-9\\4-1-6\end{bmatrix}$$

$$= \begin{bmatrix}1\\-9\\-3\end{bmatrix} = \mathbf{b}$$

NOTE: Recall from a Theorem of Sec 3.4 that the general solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding any specific solution of this system to the general solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$. Keeping in mind that the Null Space of A is the same as the soln. space of $A\mathbf{x} = \mathbf{0}$, we can rephrase that theorem in the following vector form —

THEOREM ② If $\mathbf{x}_0$ is any soln. of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for the null space of A, then every soln. of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \qquad \text{—— ③}$$

Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\mathbf{x}$ in this formula is a soln. of $A\mathbf{x} = \mathbf{b}$.

NOTE. Eqn. ③ gives a formula for the general soln. of $A\mathbf{x} = \mathbf{b}$. The vector $\mathbf{x}_0$ in that formula is called a Particular Soln. of $A\mathbf{x} = \mathbf{b}$ and the remaining part of the formula is called the General Soln. of $A\mathbf{x} = \mathbf{0}$. In words,

The General Soln. of a consistent linear system can be expressed as the sum of a Particular Soln. of that system and General Soln. of the corresponding homogeneous system.