SEC 2.1 DETERMINANTS BY COFACTOR EXPANSION

In this Section, we will define the notion of a 'Determinant'. This will enable us to give a specific formula for the inverse of an invertible matrix, whereas upto now we have had only a computational procedure for finding it. This, in turn, will eventually provide us with a formula for solutions of certain kinds of linear systems.

Recall that the 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is Invertible iff $ad-bc \neq 0$

and that the expression $ad-bc$ is called the Determinant of the matrix A, denoted by

$$\det(A) = ad-bc$$

or $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc \qquad \text{—— ①}$$

and the Inverse of A can be expressed as $$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{—— ②}$$

One of our main goals in this Chapter is to obtain an analog of formula ② that is applicable to square matrices of all orders.

MINORS AND COFACTORS:

If A is a square matrix, then the Minor of entry a_{ij} is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A, and is denoted by M_{ij}.

The no. $(-1)^{i+j}M_{ij}$ is called the Cofactor of entry a_{ij} and is denoted by C_{ij}.

Example (finding Minors and Cofactors)

Find the Minors and Cofactors of the matrix $A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 0 & 6 \\ 3 & 5 & 7 \end{bmatrix}$.

Soln.

The Minor of entry a_{11} (i.e., 1) is $M_{11} = \begin{vmatrix} 0 & 6 \\ 5 & 7 \end{vmatrix} = 0\times7 - 5\times6 = -30$

The Cofactor of a_{11} is $C_{11} = (-1)^{1+1}M_{11} = M_{11} = -30$

The Minor of entry a_{12} (i.e., -1) is $M_{12} = \begin{vmatrix} 2 & 6 \\ 3 & 7 \end{vmatrix} = 2\times7 - 6\times3 = -4$

The Cofactor of a_{12} is $C_{12} = (-1)^{1+2}M_{12} = -M_{12} = -(-4) = 4$.

The Minor of entry a_{13} (i.e., 4) is $M_{13} = \begin{vmatrix} 2 & 0 \\ 3 & 5 \end{vmatrix} = 2\times5 - 3\times0 = 10$

The Cofactor of a_{13} is $C_{13} = (-1)^{1+3}M_{13} = M_{13} = 10$

The Minor of entry a_{21} (i.e., 2) is $M_{21} = \begin{bmatrix} -1 & 4 \\ 5 & 7 \end{bmatrix} = (-1)\times7 - 5\times4 = -27$

The Cofactor of a_{21} is $C_{21} = (-1)^{2+1}M_{21} = -M_{21} = -(-27) = 27$.

The Minor of entry a_{22} (i.e., 0) is $M_{22} = \begin{vmatrix} 1 & 4 \\ 3 & 7 \end{vmatrix} = 1\times 7 - 4\times 3 = -5$

The Cofactor of a_{22} is $C_{22} = (-1)^{2+2} M_{22} = M_{22} = -5$.

Similarly, we can find the Minors and Cofactors for other entries.

<u>REMARK</u>: A minor M_{ij} and its corresponding cofactor C_{ij} are either the same or negatives of each other and the relating sign $(-1)^{i+j}$ is either $+1$ or -1 in accordance with the pattern in the 'checkerboard' array

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

for example,

$C_{11} = M_{11}$, $C_{12} = -M_{12}$, $C_{13} = M_{13}, \cdots$

$C_{21} = -M_{21}$, $C_{22} = M_{22}$, $C_{23} = -M_{23}, \cdots$

COFACTOR EXPANSIONS OF a 2X2 MATRIX

Let a 2x2 matrix is $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

The checkerboard pattern for the matrix A is $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

$\therefore$ $C_{11} = M_{11} = a_{22}$, $C_{12} = -M_{12} = -a_{21}$

$C_{21} = -M_{21} = -a_{12}$, $C_{22} = M_{22} = a_{11}$

Now $\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$ ——— ①

$$\left.\begin{aligned} &= a_{11}C_{11} + a_{12}C_{12}, \text{ from ①} \\ &= a_{21}C_{21} + a_{22}C_{22}, \text{ from ①} \\ &= a_{11}C_{11} + a_{21}C_{21}, \text{ from ①} \\ &= a_{12}C_{12} + a_{22}C_{22}, \text{ from ①} \end{aligned}\right\} \text{——— ②}$$

Each equations of ② is called a Cofactor Expansion of $\det(A)$. In each cofactor expansion, the entries and cofactors all come from the same row or same column of A. For example, in the first eqn. of ②, the entries and cofactors all come from first row of A; in the second eqn. of ②, all come from the second row of A; in the third eqn. of ②, all come from the first column of A and in the fourth, all come from second column of A.

DEFINITION OF GENERAL DETERMINANT

THEOREM. If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

This result allows us to make the following definition —

Definition: If A is an $n \times n$ matrix, then the no. obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the Determinant of matrix A. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \qquad \text{—— (1)}$$

[Cofactor expansion along the j^{th} column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \qquad \text{—— (2)}$$

[Cofactor expansion along the i^{th} row]

Example (1) Find the determinant of the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$ by cofactor expansion along the first row of A.

Soln. $\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}$

$$= 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3[(-4)\times(-2) - 4\times 3] - [(-2)\times(-2) - 5\times 3] + 0$$

$$= 3(-4) - (-11) = -1$$

Example (2) Find the determinant of the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$ by cofactor expansion along the first column of A.

Soln.

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}$$

$$= 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3[(-4)\times(-2) - 4\times 3] + 2[(-2)\times 1 - 4\times 0] + 5[1\times 3 - 0\times(-4)]$$

$$= 3(-4) + 2(-2) + 5(3) = -1.$$

REMARK: Note that in Example (2), we had to compute three cofactors whereas in Example (1), only two were needed because the third was multiplied by zero.

As a rule, the best strategy for cofactor expansion (to find determinant) is to expand along a row or column with the most zeros.

Example ③ (Smart Choice of Row or Column)

Find the determinant of the matrix, $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$.

Soln. To find det(A), it will be easiest to use cofactor expansion along the second column, since it has the most zeros:

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

Now for this 3x3 determinant, it will be easiest to use cofactor expansion along its second column, since it has most zeros:

$$\det(A) = 1 \cdot (-2) \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= -2\left[1 \times 1 - 2 \times (-1)\right]$$

$$= -6 .$$

DETERMINANT OF TRIANGULAR MATRIX

If A is an nxn Triangular matrix (upper triangular, lower triangular or diagonal), then det(A) is the product of entries on the main diagonal of the matrix

i.e., $\det(A) = a_{11}\, a_{22}\, a_{33} \cdots a_{nn}$

Example Find the determinant of the following matrix $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 5 & 1 & 0 & 5 \end{bmatrix}$.

Soln. By above rule, $\det(A) = 2 \times (-3) \times 1 \times 5 = -30$.