SEC-1.1 INTRODUCTION TO SYSTEMS OF LINEAR EQUATIONS.

LINEAR EQUATIONS. Recall that in two dimensions, a line in a rectangular xy-coordinate system can be represented by an equ. of the form

$$ax+by=c \quad (a, b \text{ not both zero})$$

and in three dimensions, a plane in a rectangular xyz-cordinate system can be represented by an equ. of the form $ax+by+cz=d \quad (a, b, c \text{ not all zero})$

Generally, we define a Linear Equation in 'n' variables $x_1, x_2, \cdots, x_n$ to be one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \qquad \text{—— (1)}$$

where $a_1, a_2, \cdots, a_n$ and b are constants and the a_i's are not all zero.

In the special case where $b=0$, Equ.(1) has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \qquad \text{—— (2)}$$

which is called a Homogeneous Linear Equ. in the variables $x_1, x_2, \cdots, x_n$.

NOTE Observe that a Linear Equ. does not involve any products or roots of variables. All variables occur only to the first power and do not appear as arguments of trigonometric, logarithmic, or exponential functions. Some of examples of linear equations are —

$$2x+3y=5, \quad x-y+z=-1, \quad x_1+x_2+\cdots+x_n=2$$

The following equations are not Linear —

$$x+3y^2=2, \quad x+y-xy=2, \quad \sin x+y=0, \quad \sqrt{x}+y+z=1$$

SYSTEM OF LINEAR EQUATIONS: A finite set of linear equations is called a System of Linear Equations. The variables are called Unknowns. for example, linear system (3) has unknowns x and y; and linear system (4) has unknowns x_1, x_2 and x_3.

$$\left.\begin{aligned} 5x+y &= 3 \\ 2x-y &= 4 \end{aligned}\right\} \qquad \text{—— (3)}$$

$$\left.\begin{aligned} 4x_1-x_2+3x_3 &= -1 \\ 3x_1+x_2+9x_3 &= -4 \end{aligned}\right\} \qquad \text{—— (4)}$$

A General Linear System of m equations in n unknowns $x_1, x_2, \cdots, x_n$ can be written as

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \qquad \text{—— (5)}$$

SOLUTION OF LINEAR SYSTEM. — A solu. of a linear system in n unknowns $x_1, x_2, ---, x_n$ is a sequence of n numbers $s_1, s_2, ---, s_n$ for which the substitution

$$x_1 = s_1, \; x_2 = s_2, \; ---, \; x_n = s_n$$ makes each equ. a true statement.

For example, the system ③ has the solu. $x=1$, $y=-2$

and the system ④ has the solu. $x_1 = 1$, $x_2 = 2$, $x_3 = -1$

These solutions can be written more succinctly as $(1,-2)$ and $(1,2,-1)$.

This notation allows us to interpret these solutions geometrically as points in two-dimensional and three-dimensional space.

More generally, a solu. $x_1 = s_1, x_2 = s_2, ---, x_n = s_n$

of a linear system in n unknowns can be written as $(s_1, s_2, ---, s_n)$ which is called an Ordered n-tuple.

If $n=2$, then the n-tuple is called an Ordered Pair and if $n=3$, then it is called an Ordered triple.

LINEAR SYSTEMS WITH TWO AND THREE UNKNOWNS. Linear systems in two unknowns arise in connection with intersection of lines. For example, consider the linear system

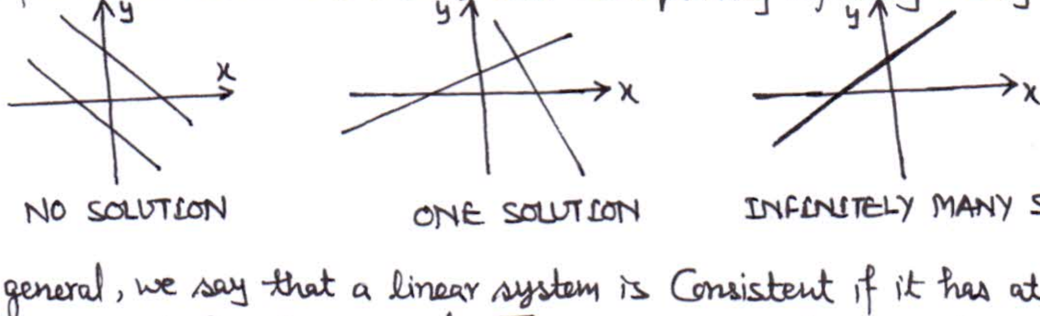
$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

in which the graph of the equations are lines in the xy-plane. Each solu. (x,y) of this system corresponds to a point of intersection of the lines, so there are three possiblities —

i) The lines may be parallel and distinct, in which case there is no intersection and consequently no solution.

ii) The lines may intersect at only one point, in which case system has exactly one solution.

iii) The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently infinitely many solutions.



NO SOLUTION ONE SOLUTION INFINITELY MANY SOLUTIONS (coincident lines)

In general, we say that a linear system is Consistent if it has at least one solution and Inconsistent if it has no solu. Thus, a Consistent linear system of two equ. in two unknowns has either one solu. or infinitely many solutions — there are no other possibilities.

The same is true for a linear system of 3 equ. in 3 unknowns

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

in which the graphs of the equations are planes. The solutions of the system, if any, correspond to points where all three planes intersect, so again we see that there are only 3 possibilities — no solu., one solu. or infinitely many solutions.

Example (A Linear System with One Solu.)

Solve the following linear system :— $\quad x-y=1 \quad$ —— ①
$\qquad 2x+y=6 \quad$ —— ②

Solu. We can eliminate x from the second equ. by adding -2 times the first equ. to the second. This yields the simplified system

$$x-y=1 \quad \text{—— ①}$$
$$3y=4 \quad \text{—— ②}'$$

From the second equ., we get $y=\frac{4}{3}$ and on substituting this value in the first equ., we get

$$x = 1+y$$

i.e; $x = 1+\frac{4}{3} \Rightarrow x=\frac{7}{3}$.

Thus, the system has the Unique Solution, $x=\frac{7}{3}$, $y=\frac{4}{3}$

Geometrically, this means that the lines represented by the equations in the system intersect at the single point $\left(\frac{7}{3}, \frac{4}{3}\right)$.

Example (A Linear System with No Solution)

Solve the following linear system :— $\quad x+y=4 \quad$ —— ①
$\qquad 3x+3y=6 \quad$ —— ②

Solu. We can eliminate x from second equ. by adding -3 times the first equ. to the second equ. This yields

$$x+y=4 \quad \text{—— ①}$$
$$0=-6 \quad \text{—— ②}'$$

The second equ. is Contradictory, so the given system has no solution.

Geometrically, this means that lines corresponding to the equ. in the system are parallel & distinct.

Example (A Linear System with Infinitely Many Solutions)

Solve the following linear system :— $\quad 4x-2y=1 \quad$ —— ①
$\qquad 16x-8y=4 \quad$ —— ②

Solu. We can eliminate x from second equ. by adding -4 times the first equ. to the second equ. This yields

$$4x-2y=1 \quad \text{—— ①}$$
$$0=0 \quad \text{—— ②}'$$

The second equ. does not impose any restrictions on x and y, and hence can be omitted. Thus, the solutions of the system are those values of x & y that satisfy single equ. $4x-2y=1$.

Geometrically, this means that lines corresponding to the equ. in the system coincide.

One way to describe the solu. set is to solve this equ. for x in terms of y to obtain

$x=\frac{1}{4}+\frac{1}{2}y$ and then assign an arbitrary value 't' to y.

This allows us to express the solu. by the pair of equations (called Parametric Equations)

$$x=\frac{1}{4}+\frac{1}{2}t \quad , \quad y=t$$

We can obtain specific numerical solutions from these equ. by substituting numerical values for the parameter.

For example, $t=0$ yields the solu., $x=\frac{1}{4}$, $y=0$

$t=1$ yields the solu., $x=\frac{3}{4}$, $y=1$

AUGMENTED MATRICES AND ELEMENTARY ROW OPERATIONS.

Let a system of m equations in n ~~unknowns~~ unknowns is

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

The Augmented Matrix for this system can be written as
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

The basic method for solving a linear system is to perform appropriate algebraic operations on the system that do not alter the solu. set and that produce a succession of increasingly simpler systems, until a point is reached where it can be ascertained whether the system is consistent and if so, what it solutions are.

The following Operations on the rows of Augmented matrix are called Elementary raw operations on a matrix ——

i) Multiply a row by a non-zero constant.

ii) Interchange any two rows.

iii) Add a constant times one row to another row.

SEC-1.2 GAUSSIAN ELIMINATION

In this Section, we will develop a systematic procedure for solving systems of linear equations. The procedure is based on the idea of performing certain operations on the rows of Augmented matrix for the system that simplifies it to a form from which the solution of the system can be ascertained by inspection.

ECHELON FORMS

Row Echelon form — A matrix is said to be in Row Echelon form if it has the following properties —

i) If a row does not consist entirely of zeros, then the first non-zero number in the row is 1. We call this a leading 1.

ii) If there are any rows that consists entirely of zeros, then they are grouped together at the bottom of the matrix.

iii) In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

Reduced Row Echelon form — A row echelon form is said to be Reduced Row Echelon form if the matrix has the following property —

iv) Each column that contains a leading 1 has zeros everywhere else in that column.

Example (Row Echelon and Reduced Row Echelon form) —

The following matrices are in Reduced Row Echelon form —

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The following matrices are in Row Echelon form but not Reduced Row Echelon form —

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

ELIMINATION METHODS TO SOLVE A SYSTEM OF LINEAR EQUATIONS.

GAUSSIAN ELIMINATION METHOD – The procedure (or algorithm) that reduces Augmented Matrix (while solving a system of linear equ.) to Raw Echelon form is called Gaussian Elimination.

Example: Find the solu. of the following system by Gaussian Elimination –

$$x_1 + x_2 + 2x_3 = 8$$
$$-x_1 - 2x_2 + 3x_3 = 1$$
$$3x_1 - 7x_2 + 4x_3 = 10$$

Solu. The given system of equations can be written as –

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$$

ie., $$A\mathbf{x} = \mathbf{b}$$

where $A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $b = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$

Now Augmented Matrix, $[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right] \quad R_2 \to (-1)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right] \quad R_3 \to R_3 + 10R_2$$

The corresponding system of equations is

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & -52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ -104 \end{bmatrix}$$

ie;
$$x_1 + x_2 + 2x_3 = 8 \quad \text{——— (i)}$$
$$x_2 - 5x_3 = -9 \quad \text{——— (ii)}$$
$$-52x_3 = -104 \quad \text{——— (iii)}$$

from equ. (iii), $x_3 = 2$

from (ii), $x_2 - 5\times 2 = -9 \Rightarrow x_2 = 1$

& from (i), $x_1 + 1 + 2\times 2 = 8 \Rightarrow x_1 = 3$

GAUSS-JORDAN ELIMINATION METHOD — The procedure (or algorithm) that reduces Augmented Matrix to Reduced Row Echelon form is called Gauss-Jordan Elimination.

Example — Solve the following system of linear equations by Gauss-Jordan Method —

$$x_1 + x_2 + 2x_3 = 8$$
$$-x_1 - 2x_2 + 3x_3 = 1$$
$$3x_1 - 7x_2 + 4x_3 = 10$$

Soln. The given system of equations can be written in matrix form as

$$Ax = b$$

where $A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $b = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$

Augmented Matrix,

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \quad R_2 \to (-1) R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} \quad R_3 \to R_3 + 10R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_3 \to \left(-\frac{1}{52}\right) R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_1 \to R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - 7R_3 \\ R_2 \to R_2 + 5R_3 \end{array}$$

The corresponding system of equations is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = 3$$
$$x_2 = 1$$
$$x_3 = 2$$

SEC - 1.3 MATRICES AND MATRIX OPERATIONS.

Matrix Notation and Terminology.

A Matrix is a rectangular array of numbers. The numbers in the array are called the Entries in the matrix.

For example, the following rectangular array with three rows and 7 columns might describe the no. of hours that a student spent studying 3 Subjects during a certain week:

	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Sun.
Math	2	3	2	4	1	4	2
History	0	3	1	4	3	2	2
Language	4	1	3	1	0	0	2

~~Column Matrix and Row Matrix — A matrix with~~

If we suppress the headings, then we are left with the following rectangular array of numbers with 3 rows and 7 columns, called a 'Matrix': —

$$\begin{bmatrix} 2 & 3 & 2 & 4 & 1 & 4 & 2 \\ 0 & 3 & 1 & 4 & 3 & 2 & 2 \\ 4 & 1 & 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

COLUMN MATRIX AND ROW MATRIX: A matrix with only one column is called a Column Vector or a Column Matrix and a matrix with only one row is called a Row Vector or a Row Matrix.

SIZE OF A MATRIX: The Size of a matrix is described in terms of the no. of rows (Horizontal lines) and columns (Vertical lines) it contains. In a size description, the first no. always denotes the no. of rows and the second denotes the no. of columns.

Example What are the sizes of the following matrices —

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}.$$

Soln. The sizes the matrices are 3×2, 1×4, 3×3, 2×1 and 1×1 respectively.

Notation: The entry that occurs in row i and column j of a matrix A will be denoted by a_{ij}. Thus a general $m \times n$ matrix is written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

When a compact notation is desired, the preceding matrix can be written as $[a_{ij}]_{m\times n}$ or simply as $[a_{ij}]$.

The entry in row i and column j of a matrix A is also commonly denoted by the symbol $(A)_{ij}$. Thus for the above matrix, we have $(A)_{ij} = a_{ij}$

NOTE. Row and column vectors are of special importance and it is common practice to denote them by boldface lowercase letters rather than capital letters. For such matrices, double subscripting of the entries is unnecessary. Thus, a general $1 \times n$ row vector '$\mathbf{a}$' and a general $m \times 1$ column vector '$\mathbf{b}$' would be written as

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots \quad a_n] \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

SQUARE MATRIX. A matrix 'A' with n rows and n columns is called a Square Matrix of order n and the entries $a_{11}, a_{22}, \cdots a_{nn}$ are said to be on the main diagonal of matrix A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

OPERATIONS ON MATRICES: So far, we have used matrices to abbreviate the work in solving systems of linear equations. For other applications, however, it is desirable to develop an 'Arithmetic of Matrices' in which matrices can be added, subtracted and multiplied in a useful way.

Equality of Matrices: – Two matrices are defined to be Equal if they have the same size and their corresponding entries are equal.

The Equality of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size can be expressed by writing $a_{ij} = b_{ij}$, where it is understood that the equalities hold for all values of i and j.

Example: Consider the matrices $A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$

If $x = 5$, then $A = B$; but for all other values of x, the matrices A & B are not equal, since not all of their corresponding entries are equal.

There is no value of x for which $A = C$, since A & C have different sizes.

Sum and Difference of Matrices: If A and B are matrices of the same size, then the Sum $A+B$ is the matrix obtained by adding the entries of matrix B to the corresponding entries of matrix A.

The Difference $A-B$ is the matrix obtained by subtracting the entries of B from the corresp. entries of A.

Matrices of different sizes cannot be added or subtracted.

In matrix notation, if $A = [a_{ij}]$ and $B = [b_{ij}]$ have same sizes, then

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij}$$

and $$(A-B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$$

Example: Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}$$

and $$C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Then $$A+B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \text{ and } A-B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

The expressions $A+C$, $B+C$, $A-C$ and $B-C$ are undefined

Scalar Multiples of Matrices: If A is any matrix and c is any scalar, then the product cA is the matrix obtained by multiplying each entry of the matrix A by const. c. The matrix cA is said to be a Scalar Multiple of A.

In matrix notation, if $A = [a_{ij}]$, then $(cA)_{ij} = c(A)_{ij} = ca_{ij}$.

Example: For the matrices $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}$, $C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$

We have $2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}$, $-B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}$, $\frac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$

Multiplication of Matrices: If A is a matrix of size $m \times r$ and B is matrix of size $r \times n$, then the Product AB is a matrix of size $m \times n$ whose entries are determined as follows —

To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together and then add up the resulting products.

Example consider the matrices $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ 0 & -1 \\ 2 & 7 \end{bmatrix}$

Find AB and BA.

Soln. Since A is a 2×3 matrix and B is a 3×2 matrix, the product AB is defined and given as —

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}_{2\times3} \begin{bmatrix} 4 & 1 \\ 0 & -1 \\ 2 & 7 \end{bmatrix}_{3\times2}$$

$$= \begin{bmatrix} 1\times4 + 2\times0 + 4\times2 & 1\times1 + 2\times(-1) + 4\times7 \\ 2\times4 + 6\times0 + 0\times2 & 2\times1 + 6\times(-1) + 0\times7 \end{bmatrix}_{2\times2} = \begin{bmatrix} 12 & 27 \\ 8 & -4 \end{bmatrix}_{2\times2}$$

Again since B is a matrix of size 3×2 and A is a matrix of size 2×3, the product BA is also defined and given as —

$$BA = \begin{bmatrix} 4 & 1 \\ 0 & -1 \\ 2 & 7 \end{bmatrix}_{3\times2} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}_{2\times3}$$

$$= \begin{bmatrix} 4\times1 + 1\times2 & 4\times2 + 1\times6 & 4\times4 + 1\times0 \\ 0\times1 + (-1)\times2 & 0\times2 + (-1)\times6 & 0\times4 + (-1)\times0 \\ 2\times1 + 7\times2 & 2\times2 + 7\times6 & 2\times4 + 7\times0 \end{bmatrix}_{3\times3} = \begin{bmatrix} 6 & 14 & 16 \\ -2 & -6 & 0 \\ 16 & 46 & 8 \end{bmatrix}_{3\times3}$$

MATRIX FORM OF A LINEAR SYSTEM OF EQUATIONS: Matrix multiplication has an imp. application to system of linear equations. Consider a system of m linear equations in n unknowns —

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}$$

Since two matrices are equal iff their corresp. entries are equal, we can replace the m equations in this system by the single matrix equ.

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

i.e., $$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If we designate these matrices by A, $\mathbf{x}$ and $\mathbf{b}$ resp., then we can replace the original system of m equations in n unknowns has been replaced by the single matrix equ.

$$A\mathbf{x} = \mathbf{b}$$

The matrix A in this equ. is called the Coefficient Matrix of the system. The Augmented Matrix for the system is obtained by adjoining '$\mathbf{b}$' to A as the last column; thus the Augmented Matrix is

$$[A|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

TRANSPOSE OF A MATRIX: If A is any $m \times n$ matrix, then the Transpose of A is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A and is denoted by A^T.

Example: find the transposes of the following matrices —

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}, \quad D = [4]$$

Soln.

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \ \& \ D^T = [4].$$

TRACE OF MATRIX: If A is a square matrix, then the Trace of A, is defined to be the sum of the entries on the main diagonal of A and is denoted by $tr(A)$. The trace of A is undefined if A is not a square matrix.

Example find the traces of the following matrices —

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

Soln.

$$tr(A) = a_{11} + a_{22} + a_{33}$$

$$tr(B) = -1 + 5 + 7 + 0 = 11.$$

SEC-1.4 INVERSES; ALGEBRAIC PROPERTIES OF MATRICES

In this Section, we will discuss some of the algebraic properties of matrix operations. We will see that the basic rules of arithmetic for real numbers hold for matrices, but we will also see that some do not hold.

PROPERTIES OF MATRIX ARITHMETIC: Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

i) $A+B = B+A$ [Commutative Law for Addition]

ii) $A+(B+C) = (A+B)+C$ [Associative Law for Addition]

iii) $A(BC) = (AB)C$ [Associative Law for Multiplication]

iv) $A(B+C) = AB+AC$ [Left Distributive Law]

v) $(B+C)A = BA+CA$ [Right Distributive Law]

vi) $A(B-C) = AB-AC$

vii) $(B-C)A = BA-CA$

viii) $a(B+C) = aB+aC$

ix) $a(B-C) = aB-aC$

x) $(a+b)C = aC+bC$

xi) $(a-b)C = aC-bC$

xii) $a(bC) = (ab)C$

xiii) $a(BC) = (aB)C = B(aC)$

Example ① Associativity of Matrix Multiplication

As an illustration of the Associative law for matrix multiplication, consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \quad \& \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

Then
$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix}$$

$$BC = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}$$

Now
$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

and
$$(AB)C = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

Hence $A(BC) = (AB)C$

Question: Verify Associative Law of Matrix Multiplication by Example.

PROPERTIES OF MATRIX MULTIPLICATION:

We know that in real arithmetic, it is always true that $ab = ba$, which is called the Commutative Law for Multiplication. But in Matrix Arithmetic, the Equality of AB and BA can fail for three possible reasons —

(i) AB may be defined and BA may not (for example, if A is 2×3 and B is 3×4)

(ii) AB and BA may both be defined but they may have different sizes (for example, if A is 2×3 and B is 3×2)

(iii) AB and BA both be defined and have the same size but the two matrices may be different (as illustrated in the next example).

Example (Order Matters in Matrix Multiplication)

Consider the matrices $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

Multiplication gives
$$AB = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}$$

and
$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

Thus AB and BA have same sizes but $AB \neq BA$.

ZERO MATRICES: A matrix whose entries are all zero, is called a Zero Matrix.

Some examples are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $[0]$

We will denote a zero matrix by $\mathbf{0}$ unless it is important to specify its size, in which case we will denote the $m \times n$ zero matrix by $0_{m \times n}$.

PROPERTIES OF ZERO MATRICES. If c is a scalar and if the sizes of the matrices are such that the operations can be performed, then

i) $A + \mathbf{0} = \mathbf{0} + A = A$

ii) $A - \mathbf{0} = A$

iii) $A - A = A + (-A) = \mathbf{0}$

iv) $0A = \mathbf{0}$

v) If $cA = \mathbf{0}$, then $c = 0$ or $A = \mathbf{0}$.

NOTE: Since we know that commutative law of multiplication of real arithmetic is not valid in matrix arithmetic, it should not be surprising that there are other rules that fail as well. For example, consider the following two laws of real arithmetic–

i) If $ab = ac$ and $a \neq 0$, then $b = c$ [The Cancellation Law]

ii) If $ab = 0$, then at least one of the factors on the left is 0.

The next two examples show that these laws are not universally true in Matrix Arithmetic.

Example ① Failure of Cancellation Law

Consider the matrices $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

Now $AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

$AC = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

Thus $AB = AC$ ——— ①

Although $A \neq \mathbf{0}$, canceling A from both sides of ①, we get $B = C$, which is incorrect conclusion. Thus, the cancellation law does not hold, in general, for matrix multiplication.

Example ② Zero Product with Non-Zero Matrices

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$

Here $A \neq \mathbf{0}$, $B \neq \mathbf{0}$ but

$AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

i.e. $AB = \mathbf{0}$

IDENTITY MATRICES : A square matrix with 1's on the main diagonal and zeros elsewhere, is called an Identity Matrix.

Some examples are $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

An Identity Matrix is denoted by the letter I. If it is important to emphasize the size, we will write I_n for the $n \times n$ identity matrix.

NOTE. To explain the role of identity matrices in matrix arithmetic, let us consider the effect of multiplying a general 2×3 matrix A on each side by an identity matrix.

Multiplying on the right by 3×3 identity matrix yields

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

and multiplying on the left by 2×2 identity matrix yields

$$I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

The same result holds in general; that is, if A is any $m\times n$ matrix, then

$$AI_n = A \quad \text{and} \quad I_m A = A$$

THEOREM. If R is the Reduced row echelon form of an $n\times n$ matrix A, then either R has a row of zeros or R is the identity matrix I_n.

INVERSE OF A MATRIX: If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be Invertible (or Non-singular) and B is called an Inverse of A. If no such matrix can be found, then matrix A is said to be Singular.

REMARK. The relationship $AB = BA = I$ is not changed by interchanging A & B, so if A is Invertible and B is an Inverse of A, then it is also true that B is Invertible, and A is an Inverse of B.

Thus, when $AB = BA = I$

we say that A and B are Inverses of each other.

Example **An Invertible Matrix**

Let $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

Then $AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Thus, A and B are Invertible and each is an inverse of the other.

PROPERTIES OF INVERSES:- It is reasonable to ask whether an invertible matrix can have more than one inverse. The next theorem shows that the answer is No —

An invertible matrix has exactly one Inverse.

THEOREM. If B & C are both inverses of matrix A, then $B = C$.

Proof. Since B is an inverse of A, so $BA = I$

Multiplying both sides on right by C gives $(BA)C = IC$

i.e., $(BA)C = C$ —— ①

We know that $(BA)C = B(AC)$

$= BI$

$(BA)C = B$ —— ②

From ① & ②, we have $B = C$.

NOTE. As a consequence of this important result, we can now speak of 'the' inverse of an invertible matrix.

If A is invertible, then its inverse will be denoted by A^{-1}.

Thus, $AA^{-1} = I$ and $A^{-1}A = I$.

We give the following Theorem that specifies conditions under which a 2x2 matrix is invertible and provides a simple formula for its inverse. We will develop a method for computing the inverse of an invertible matrix of any size in next Section.

<u>THEOREM</u>. The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is Invertible iff $ad-bc \neq 0$, in which case the Inverse is given by $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

<u>NOTE</u> The quantity $ad-bc$ is called the Determinant of 2x2 matrix A and is denoted by $\det(A) = ad-bc$

or alternatively by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$.

Thus, the above Theorem states that a 2x2 matrix A is Invertible if and only if its determinant is non-zero and if invertible, then its inverse can be obtained by above formula.

<u>Example</u>: Determine whether the following matrices are Invertible. If so, find their Inverses.

(i) $A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ (ii) $B = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

<u>Solu.</u> (i) $\det(A) = 6\times 2 - 1\times 5 = 7$ (non-zero)

Thus matrix A is Invertible and its inverse is

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

We can confirm that $AA^{-1} = A^{-1}A = I$

(ii) $\det(B) = (-1)\times(-6) - 3\times 2 = 0$

Thus matrix B is not Invertible.

<u>Example</u>: <u>Solu. of a Linear System by Matrix Inversion</u>

INVERSE OF MATRIX PRODUCTS

THEOREM: If A & B are Invertible matrices with the same size, then AB is Invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

This Result can be extended to three or more matrices.

ie; $$(A_1A_2\cdots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1}\cdots A_2A_1$$

Example: Verify $(AB)^{-1} = B^{-1}A^{-1}$ by taking an example.

Soln. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$

Now $$AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}$$

$$\det(AB) = 7\times 8 - 9\times 6 = 2$$

$$\therefore (AB)^{-1} = \frac{1}{2}\begin{bmatrix} 8 & -6 \\ -9 & 7 \end{bmatrix}$$

$$(AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix} \quad \text{------ (1)}$$

Again $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

$\det(A) = 1\times 3 - 2\times 1 = 1$

$\therefore A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$

$\det(B) = 3\times 2 - 2\times 2 = 2$

$B^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}$

Now $$B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

ie; $$B^{-1}A^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix} \quad \text{------ (2)}$$

$\therefore$ From (1) & (2), $(AB)^{-1} = B^{-1}A^{-1}$.

SEC-1.5 ELEMENTARY MATRICES AND A METHOD FOR FINDING A^{-1}

In this Section, we will develop an algorithm for finding the inverse of a matrix and we will discuss some of the basic properties of invertible matrices.

In Section 1.1, we defined three elementary row operations on a matrix A —

(i) Multiply a row by a non-zero constant c.

(ii) Interchange two rows.

(iii) Add a const. c times one row to another.

It should be evident that if we let B be the matrix that results from A by performing one of the operations in this list, then the matrix A can be recovered from B by performing the corresponding operation in the following list —

(i) Multiply the same row by $\frac{1}{c}$.

(ii) Interchange the same two rows.

(iii) If B resulted by adding c times row r_1 of A to row r_2, then add $-c$ times r_1 to r_2.

It follows that if B is obtained from A by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to B recovers A

ROW EQUIVALENT MATRICES — Matrices A and B are said to be Row Equivalent if either (hence each) can be obtained from the other by a sequence of elementary row operations.

ELEMENTARY MATRIX — An nxn matrix is called an Elementary Matrix if it can be obtained from nxn identity matrix I_n by performing a single elementary row operation.

Example (Elementary Matrices and Row Operations)

Listed below are four elementary matrices and the operations that produce them —

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \leftarrow \text{Multiply the second row of } I_2 \text{ by } -3.$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{Add three times the third row of } I_3 \text{ to the first row.}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \leftarrow \text{Interchange the second and fourth rows of } I_4$$

THEOREM (Row Operations by Matrix Multiplication) – If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is matrix that results when same row operation is performed on A.

Example: (Using Elementary Matrices) –

Consider the matrix
$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

and consider elementary matrix
$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

which results from adding 3 times the first row of I_3 to the third row.

The Product EA is
$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

which is same matrix that results when we add 3 times the first row of A to the third row.

TABLE

Row Operation on I that Produces E	Row Operation on E that produces I
i) Multiply row i by $c \neq 0$	Multiply row i by $\frac{1}{c}$
ii) Interchange rows i and j	Interchange rows i and j
iii) Add c times row i to row j	Add $-c$ times row i to row j

Example (Row Operations and Inverse Row Operations) – In each of the following, an elementary row operation is applied to 2×2 Identity matrix I to obtain an Elementary matrix E, then E is restored to I by applying inverse row operation.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[\text{Multiply second row by } 7]{} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \xrightarrow[\text{Multiply second row by } \frac{1}{7}]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[\text{Interchang first \& second rows}]{} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow[\text{Interchange first \& second rows}]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[\substack{\text{Add 5 times the second row} \\ \text{to the first row}}]{} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \xrightarrow[\substack{\text{Add } -5 \text{ times the second row} \\ \text{to the first row}}]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

THEOREM. Every Elementary matrix is Invertible and the Inverse is also an Elementary matrix.

THEOREM (EQUIVALENT STATEMENTS)

If A is an $n \times n$ matrix, then the following statement are equivalent (that is, all are true or all are false) —

(i) A is Invertible.

(ii) $A\mathbf{x} = \mathbf{0}$ has only trivial (or zero) solution.

(iii) The reduced row echelon form of A is I_n.

(iv) A is expressible as a product of elementary matrices.

NOTE. The validity of any one statement implies the validity of all the others, and the falsity of any one implies the falsity of the others.

INVERSION ALGORITHM. —

To find the Inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to the Identity matrix and then perform the same sequence of operations on I_n to obtain A^{-1}.

A simple method for carrying out this procedure is given in following example —

Example Using Row Operations find the inverse of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

Soln. Let the partitioned matrix of the form $[A \mid I]$

Then we will apply row operations to this matrix until A is reduced to I; these operations will convert I to A^{-1} so the final matrix has the form $[I \mid A^{-1}]$

The Computations are as follows —

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \quad R_3 \to R_3 + 2R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \quad R_3 \to (-1)R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 + 3R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \quad R_1 \to R_1 - 2R_2$$

$$\sim \left[I \mid A^{-1} \right]$$

Thus $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

TO SHOW THAT A MATRIX IS NOT INVERTIBLE —

often it will not be known in advance if a given $n \times n$ matrix A is invertible. However, if it is not, then it will be impossible to reduce A to I_n by elementary row operations. This will be signaled by a row of zeros appearing on the left side of the partition at some stage of Inversion Algorithm. If this occurs, then we can stop the computations and conclude that matrix A is not Invertible.

Example (Showing that a Matrix is not Invertible)

Show that the matrix A is not Invertible where $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

Soln. Here $[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \quad R_3 \to R_3 + R_2$$

Since we have obtained a row of zeros on left side, the matrix A is not Invertible.

ANALYZING HOMOGENEOUS SYSTEMS — A Homogeneous Linear System of Equ. has only Trivial Solu. iff its coefficient matrix is Invertible.

Example: Determine whether the given Homogeneous System has non-trivial solutions —

(i)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 \quad + 8x_3 &= 0 \end{aligned}$$

(ii)
$$\begin{aligned} x_1 + 6x_2 + 4x_3 &= 0 \\ 2x_1 + 4x_2 - x_3 &= 0 \\ -x_1 + 2x_2 + 5x_3 &= 0 \end{aligned}$$

Solu.

(i) The Coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

which is Invertible by previous example.
Hence, the system of equ. (i) has only trivial solu.

(ii) The Coefficient matrix is $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

which is not Invertible by previous example.
Hence, the system of equ. (ii) has non-trivial solu.

SEC-1.6 MORE ON LINEAR SYSTEMS AND INVERTIBLE MATRICES.

SOLVING LINEAR SYSTEMS BY MATRIX INVERSION – Thus far we have studied two procedures for solving Linear Systems – Gauss-Jordan Elimination and Gaussian Elimination. The following Theorem provides an actual formula for the soln. of a linear system of n eqn's in n unknowns in the case where coeff. matrix is Invertible.

THEOREM : If A is an Invertible matrix of size $n \times n$, then for each $n \times 1$ matrix $\mathbf{b}$, the system of eqn. $A\mathbf{x} = \mathbf{b}$ has exactly one soln., namely, $\mathbf{x} = A^{-1}\mathbf{b}$.

Example (Soln. of a Linear System Using A^{-1}).

Find the soln. of a system of linear equations:–

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}$$

Soln. In matrix form, the given system can be written as $Ax = b$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$

In example of preceding Section, we showed that A is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Now the soln. of the system is

$$\begin{aligned} x &= A^{-1}b \\ &= \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \end{aligned}$$

NOTE. Keep in mind that this method only applies when the system has as many equations as unknowns and the coefficient matrix is Invertible.

PROPERTIES OF INVERTIBLE MATRICES. – Up to now, to show that an $n\times n$ matrix A is invertible, it has been necessary to find an $n\times n$ matrix B such that $AB = I$ & $BA = I$

The next Theorem shows that if we produce an $n\times n$ matrix B satisfying either condition, then the other condition holds automatically.

THEOREM. Let A be a square matrix

i) If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$

ii) If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

SEC-1.7 DIAGONAL, TRIANGULAR AND SYMMETRIC MATRICES

In this Section, we will discuss matrices that have various special forms. These matrices arise in a wide variety of applications and will also play an important role in our subsequent work.

DIAGONAL MATRICES.

A square matrix in which all entries off the main diagonal are zero, is called Diagonal Matrix. Some of the examples of Diagonal matrices are given as —

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

A general $n \times n$ diagonal matrix D can be written as —

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix} \quad \text{— (1)}$$

Inverse of Diagonal Matrix

A Diagonal Matrix is Invertible iff all of its diagonal entries are non-zero. In this case, the inverse of (1) is

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & 0 & \cdots & 0 \\ 0 & 1/d_2 & 0 & \cdots & 0 \\ 0 & 0 & 1/d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1/d_n \end{bmatrix} \quad \text{— (2)}$$

We can confirm formula (2) by showing that $DD^{-1} = D^{-1}D = I$.

Power of Diagonal Matrix

If D is diagonal matrix given by (1) and k is a positive integer, then

$$D^k = \begin{bmatrix} d_1^k & 0 & 0 & \cdots & 0 \\ 0 & d_2^k & 0 & \cdots & 0 \\ 0 & 0 & d_3^k & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n^k \end{bmatrix} \quad \text{— (3)}$$

Example: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, then find A^{-1}, A^5 and A^{-5}.

Soln. By formula (2), $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$

By formula (3), $A^5 = \begin{bmatrix} (1)^5 & 0 & 0 \\ 0 & (-3)^5 & 0 \\ 0 & 0 & (2)^5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix}$

Now $A^{-5} = (A^5)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/243 & 0 \\ 0 & 0 & 1/32 \end{bmatrix}$

TRIANGULAR MATRICES – A square matrix in which all the entries above the main diagonal are zero, is called Lower Triangular Matrix and a square matrix in which all the entries below the main diagonal are zero, is called Upper Triangular Matrix.

A matrix that is either upper triangular or lower triangular is called Triangular Matrix.

Example: Upper and Lower Triangular Matrices

A general 4x4 upper triangular Matrix $\rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$

A general 4x4 lower triangular matrix $\longrightarrow \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$

NOTE. observe that diagonal matrices are both upper triangular and lower triangular since they have zeros below and above the main diagonal. Observe also that a square matrix in row echelon form is Upper Triangular since it has zoes below the main diagonal.

PROPERTIES OF TRIANGULAR MATRICES.

i) A square matrix $A = [a_{ij}]$ is Upper Triangular iff all entries to the left of the main diagonal are zero, that is, $a_{ij} = 0$ if $i > j$.

ii) A square matrix $A = [a_{ij}]$ is Lower Triangular iff all entries to the right of the main diagonal are zero, that is, $a_{ij} = 0$ if $i < j$.

iii) A square matrix $A = [a_{ij}]$ is Upper Triangular iff i^{th} row starts with atleast $i-1$ zeros $\forall i$.

iv) A square matrix $A = [a_{ij}]$ is Lower Triangular iff j^{th} column starts with atleast $j-1$ zeros for each j.

THEOREM

(i) The Transpose of a Lower Triangular Matrix is Upper Triangular Matrix and the Transpose of an Upper Triangular Matrix is Lower Triangular Matrix.

(ii) The Product of Lower Triangular Matrices is Lower Triangular Matrix and the Product of Upper Triangular Matrices is Upper Triangular Matrix.

(iii) A Triangular Matrix is Invertible iff its diagonal entries are all non-zero.

(iv) The Inverse of an Invertible Lower Triangular Matrix is Lower Triangular and the Inverse of an Invertible Upper Triangular Matrix is Upper Triangular.

SYMMETRIC MATRICES. —

A square matrix 'A' is said to be Symmetric if $A^T = A$.

It is easy to recognize a Symmetric matrix by inspection: The entries on the main diagonal have no restriction, but mirror images of entries across the main diagonal must be equal.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 5 \\ 3 & 5 & 4 \end{bmatrix}$$

Example (Symmetric Matrices) — The following matrices are Symmetric since each is equal to its own transpose —

$$\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 5 \\ 3 & 5 & 4 \end{bmatrix}, \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

ALGEBRAIC PROPERTIES OF SYMMETRIC MATRICES —

THEOREM. If A & B are Symmetric matrices with the same size and k is any scalar, then

i) A^T is Symmetric.

ii) $A+B$ and $A-B$ are Symmetric.

iii) kA is Symmetric.

PRODUCT OF TWO SYMMETRIC MATRICES. It is not true, in general, that the product of Symmetric matrices is Symmetric.

THEOREM: The product of two symmetric matrices is symmetric iff the matrices commute.

Proof. Let A & B are symmetric matrices with same size, then $A^T = A$ & $B^T = B$

Now $(AB)^T = B^T A^T$

ie, $(AB)^T = BA$

Thus $(AB)^T = AB$ iff $AB = BA$, that is, iff A & B commute to each other.

INVERTIBILITY OF SYMMETRIC MATRICES — In general, a symmetric matrix need not be invertible. For example, a diagonal matrix with a zero on the main diagonal is Symmetric but not invertible. However, the following theorem shows that if a symmetric matrix happens to be invertible, then its INVERSE must also be symmetric.

THEOREM. If A is an Invertible Symmetric matrix, then A^{-1} is Symmetric.

Proof. Let A is invertible and symmetric, then $A^T = A$

Now $(A^{-1})^T = (A^T)^{-1} = (A)^{-1}$

i.e., $(A^{-1})^T = A^{-1}$

$\Rightarrow$ A^{-1} is Symmetric.

PRODUCTS AA^T and A^TA : Matrix products of the form AA^T and A^TA arise in a variety of applications. If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, so the products AA^T and A^TA are both square matrices — the matrix AA^T has size $m \times m$ and the matrix A^TA has size $n \times n$.

Such products are always Symmetric since

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

and $$(A^TA)^T = A^T(A^T)^T = A^TA$$

Example (The Product of a Matrix and its Transpose is Symmetric)

Let A is a 2×3 matrix given by $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$

Now $$AA^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$, which is a Symmetric matrix.

Again, $$A^TA = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$, which is also Symmetric.

NOTE: Later in the text, we will obtain general conditions under which AA^T and A^TA are Invertible. However, in the special case when A is square, we have the following result —

THEOREM. If A is an Invertible matrix, then AA^T and A^TA are also Invertible.