Ministry of Higher Education
Kingdom of Saudi Arabia

CSTS
SEU, KSA

Discrete Mathematics (Math 150)
Level III, Assignment 3
(2015)

1. State whether the following statements are true or false: [9]

   (a) If a mathematical statement $P(n)$ is true for all $n \in \mathbb{Z}^+ \cup \{0\}$, then $P(1)$ will be the basis step in the principle of mathematical induction.

   (a) False

   (b) In the principle of mathematical induction, the inductive step is equivalent to the conditional statement $\forall k\ (P(k) \rightarrow P(k+1))$.

   (b) True

   (c) The recursive definition of the set $A = \{1, 6, 11, 16, 21, \dots\}$ is $1 \in A$; $x \in A \rightarrow x + 5 \in A$..

   (c) True

   (d) There are 24 ways by which three digits number can be formed with the digits 7, 4, 1 and 2.

   (d) False

   (e) $C(n, r) = C(n, n - r)$.

   (e) True

   (f) The value of $P(5, 3)$ is 120.

   (f) False

   (g) The recurrence relation $a_n = 2a_{n-1} + 3a_{n-4} - 6a_{n-3} + 4$ is homogeneous.

   (g) False

   (h) The characteristic root of the recurrence relation $a_n = 2a_{n-1}$ is real.

   (h) True

   (i) The recurrence relation $a_n = a_{n-1} + 3a_{n-4} - 8$ is not linear.

   (i) False

2. Select one of the alternatives from the following questions as your answer. [9]

(a) The sums of the first n positive odd integers are

A. $2n+1$

B. $n^2(n-1)$

C. n^2

D. $(n-1)(n+1)$

(b) Let $P(n)$ be a mathematical statement and let $P(n) \to P(n+1)$ for all natural numbers, then $P(n)$ is true

A. for all $n > 1$.

B. for all $n > m$, m being a fixed positive integer.

C. for all n.

D. Nothing can be said.

(c) Let $P(n) : 2^n < n!$, where n is a natural number, then $P(n)$ is true

A. for all n.

B. for all $n > 2$.

C. for all $n > 3$.

D. None of the above.

(d) Which of the following is equivalent to ${}^9C_6$?

A. $\dfrac{9!}{6!3!}$

B. ${}^9C_6$

C. $\frac{P(9,6)}{6!}$

D. All of the above.

(e) The number of arrangements that can be made with the letters of the word MISSISSIPPI are

A. $\dfrac{11!}{4!4!2!}$

B. $\dfrac{11!}{4!4!}$

C. $\dfrac{4!4!2!}{11!}$

D. $\dfrac{11!}{4!2!}$

(f) The coefficient of x^8y^7 in the expansion of $(7x-4y)^{15}$ is

A. $\left(\begin{array}{c} 15 \\ 8 \end{array} \right) 7^8 4^7$

B. $-\begin{pmatrix} 15 \\ 7 \end{pmatrix} 7^8 4^7$

C. $-\begin{pmatrix} 15 \\ 8 \end{pmatrix} 7^8 4^7$

D. $\begin{pmatrix} 15 \\ 7 \end{pmatrix}$

(g) Which of the following recurrence relation have degree 3?

A. $a_n = 3a_{n-1} + a_{n-3} - 13a_{n-4} + 3$

B. $a_n = 6a_{n-1} + a_{n-4}4a_{n-3}$

C. $a_n = -2a_{n-2} + 5a_{n-3} - 3a_{n-1} + 9$

D. B and C both.

(h) The characteristic roots of the recurrence relation $a_n = -4a_{n-1} - 4a_{n-2}$ are

A. 2, -2

B. -2, -2

C. 1, 2

D. 2, 3

(i) The characteristic equation of the recurrence relation $a_n = -3a_{n-2} + 4a_{n-3}$ is

A. $r^3 - 3r - 4 = 0$

B. $r^3 + 3r + 4 = 0$

C. $r^3 + 3r - 4 = 0$

D. $r^3 - 3r + 4 = 0$

3. Prove by principle of mathematical induction, for all positive integers n, that [2]

$$1 + 3 + 3^2 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}$$

Solution: Let $P(n): \quad 1 + 3 + 3^2 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}$

We will prove it by PMI.

- **Basis Step:** We will show that $P(1)$ is true.

  $P(1): \quad 3^{1-1} = \frac{3-1}{2}$

  1=1.

  $\Rightarrow \quad P(1)$ is true.

- **Inductive Step:** Suppose $P(k)$ is true. *i.e.*
  $P(k): \quad 1+3+3^2+\cdots+3^{k-1}=\dfrac{3^k-1}{2}$

  Now we will show that
  $P(k+1): \quad 1+3+3^2+\cdots+3^{k-1}+3^k=\dfrac{3^{k+1}-1}{2}$ is true.

$$\begin{aligned}L.H.S &= 1+3+3^2+\cdots+3^{k-1}+3^k\\ &= \frac{3^k-1}{2}+3^k\\ &= \frac{3^k(1+2)-1}{2}\\ &= \frac{3^{k+1}-1}{2}\\ &= R.H.S.\end{aligned}$$

$\Rightarrow \quad P(k+1)$ is true.

$\Rightarrow$ By PMI, given mathematical statement is true for all $n \in \mathbb{N}$.

4. Find the value of $f(5)$, if f is defined recursively by $f(0)=f(1)=1$ and for $n=1,2,3,\ldots$ [2]
   $f(n+1)=\dfrac{f(n}{f(n-1)}$.

   Solution: Given that

$$\begin{aligned}f(n+1) &= \frac{f(n}{f(n-1)}\\ f(2) &= \frac{f(1)}{f(0)}=1\\ f(3) &= \frac{f(2)}{f(1)}=1\\ f(4) &= \frac{f(3)}{f(2)}=1\\ f(5) &= \frac{f(4)}{f(3)}=1.\end{aligned}$$

5. If ${}^nC_r$ represents the number of combinations of n items taken r at a time, what is the value of $\sum_{r=1}^{3} {}^nC_r$ when $n=4$? [2]

Solution: Given that $n = 4$, so

$$\begin{aligned}\sum_{r=1}^{3} {}^4C_r &= {}^4C_1 + {}^4C_2 + {}^4C_3 \\ &= \frac{4!}{3!} + \frac{4!}{2!2!} + \frac{4!}{3!} \\ &= 4 + 6 + 14 \\ &= 14.\end{aligned}$$

Therefore $\sum_{r=1}^{3} {}^4C_r = 14$.

6. Find the number of subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ having 5 elements.

Solution: Here the order of choosing the elements doesn't matter and this is a problem in combinations.

We have to find the number of ways of choosing 5 elements of this set which has 12 elements. This can be done in:

$$^{12}C_5 = \frac{12!}{5!7!} = 792 \text{ ways.}$$

7. Solve the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$. [2]

Solution: The corresponding characteristic equation of the given recurrence relation is given by
$r^2 - 7r + 10 = 0$, which is a quadratic equation having roots 2 and 5.
Therefore the solution of the recurrence relation is given by

$$a_n = c_1 2^n + c_2 5^n$$

where $c_1,\ c_2$ are some coefficient.

8. Determine which of these are linear homogeneous recurrence relation with constant coefficient. Also, find the degree of those that are. [2]

   1. $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$
   2. $a_n = a_{n-2} + 5a_{n-3}^2$

3. $a_n = -2a_{n-3} + 4a_{n-4}^{1/2} + 6$
4. $a_n = 9a_{n-5} + 3a_{n-2} + a_{n-1}$

Solution: 1 and 4 recurrence relation are LHRR, while 2 and 3 relation are not linear.
degree of recurrence relation $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$ is 3.
degree of recurrence relation $a_n = 9a_{n-5} + 3a_{n-2} + a_{n-1}$ is 5.