RELATIONS

CHAPTER - 9

WEEK - 12

BINARY RELATION:- A binary relation from set A to set B is a subset of $A \times B$

If $A = \{a, p, x\}$, $B = \{b, q, y\}$

$A \times B = \{(a,b), (a,q), (a,y), (p,b), (p,q), (p,y), (x,b), (x,q), (x,y)\}$

As $R \subset A \times B \Rightarrow R = \{(a,b), (p,q), (x,y)\}$.

When $(a,b) \in R$, we use the notation $a R b$

When $(a,b) \in R$, we use the notation $a \not{R} b$.

For the above example.

R	b	q	y
a	✓		
p		✓	
x			✓

RELATION ON A SET: A relation on a set A is a relation from A to A.

(or)

Relation on a set A is a subset of $A \times A$

Examples :-

1) Consider a relation on a set $A = \{1, 2, 3, 4\}$.
defined by $R = \{(a, b) \mid a \text{ divides } b\}$, then

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4) \\ (3,3), (4,4)\}.$$

2) The number of relations on a set with n elements is 2^{n^2}

PROPERTIES OF RELATIONS :-

REFLEXIVE : A relation R on a set A is said to be reflexive if $(a, a) \in R$ for every $a \in A$

Examples :- Consider the following relations on $A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\} \to$ NOT REFLEXIVE

$R_2 = \{(1,1), (1,2), (2,1)\}$ — Not Reflexive.

$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$ — Reflexive.

$R_4 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$ — Reflexive.

2) Let R be a relation on the set of integers given by $R = \{(a, b) \mid a \leq b\}$. Then R is reflexive.

3) Let R be a relation on the set of integers given by $R = \{(a, b) \mid a = b\}$. Then R is reflexive.

4. "DIVIDES" relation on the set of Positive integers that is $R = \{(a,b) \mid a \text{ divides } b\}$ where $a, b \in Z^+$ is reflexive

5. "DIVIDES" relation on the set of integers that is $R = \{(a,b) \mid a \text{ divides } b \text{ where } a, b \in Z\}$. is not reflexive (Reason: 0 does not divide 0).

SYMMETRIC RELATION:- A relation R on a set A is said to be symmetric if $(b,a) \in R$ whenever $(a,b) \in R$ for all $a, b \in A$.

ANTISYMMETRIC RELATION:- A relation R on a set A is said to be antisymmetric if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$ for all $a, b \in A$.

NOTE:- 1) A relation is symmetric if and only if

$$a R b \Rightarrow b R a$$

2) A relation is Antisymmetric if and only if there are no pairs of distinct elements a and b with aRb and bRa

3) The terms symmetric and antisymmetric are not opposites.

4) A relation can be both symmetric and antisymmetric

Eg $R = \{(a,b) / a = b\}$ on the set of integers.

5) A relation cannot be both symmetric and antisymmetric if it contains some pair of distinct elements i.e. (a,b). Where $a \neq b$.

TRANSITIVE RELATION :- A relation R on a set A is said to be transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ for all $a,b,c \in A$

Examples. 1) $A = \{1,2,3,4\}$.

$R = \{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$.

R is not reflexive, since $(1,1),(4,4) \notin R$

R is not symmetric, since $(2,4) \in R$ and $(4,2) \in R$

R is not antisymmetric.

R is transitive

2) Let $A = \{1,2,3,4\}$

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$.

$R_2 = \{(1,1),(1,2),(2,1)\}$

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$

$R_4 = \{(2,1),(3,1),(3,2),(4,1)\ (4,2),(4,3)\}$.

$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

$R_6 = \{(3,4)\}$.

R_3, R_5 are Reflexive

R_2, R_3 are Symmetric

R_4, R_5, R_6 are antisymmetric.

R_4, R_5, R_6 are transitive

3) Consider the following relations on the set of integers.

$R_1 = \{(a,b) \mid a \leq b\}$

$R_2 = \{(a,b) \mid a > b\}$

$R_3 = \{(a,b) \mid a = b \text{ (or) } a = -b\}$

$R_4 = \{(a,b) \mid a = b\}$.

$R_5 = \{(a,b) \mid a = b+1\}$

$R_6 = \{(a,b) \mid a+b \leq 3\}$.

R_1, R_3, R_4 are reflexive

R_3, R_4, R_6 are symmetric

R_1, R_2, R_4 and R_5 are antisymmetric

R_1, R_2, R_3 and R_4 are transitive.

Combining Relation :-

$A = \{1, 2, 3\}$; $B = \{1, 2, 3, 4\}$

$R_1 = \{(1,1), (2,2), (3,3)\}$

$R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$

$R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2), (3,3)\}$.

$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

Composition

R is a relation from A to B

~~R~~ S is a relation from B to C.

The composition of R and S is denoted by $S \circ R$ is a relation from A to C

Where $(a,b) \in R$ and $(b,c) \in S \Rightarrow (a,c) \in S \circ R$.

Example :- R is a relation from $\{1,2,3\}$ to $\{1,2,3,4\}$

S is a relation from $\{1,2,3,4\}$ to $\{0,1,2\}$.

$R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$

$S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}$

$S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$.

NOTE

$$R^2 = R \circ R$$

$$R^3 = R^2 \circ R = (R \circ R) \circ R$$

$$R = \{(1,1), (2,1), (3,2), (4,3)\}$$

$$R^2 = R \circ R = \{(1,1), (2,1), (3,1), (4,2)\}.$$

$$R^3 = R^2 \circ R = \{(1,1), (2,1), (3,2), (4,3)\} \circ \{(1,1), (2,1), (3,1), (4,2)\}$$

$$= \{(1,1), (2,1), (3,1), (4,1)\}.$$

REPRESENTING RELATIONS :-

Representing relations using Matrices (Zero one Matrices)

If R is a relation from $A = \{a_1, a_2, \ldots a_m\}$ to $B = \{b_1, b_2, \ldots b_n\}$. The relation R can be represented by a matrix

$$M_R = [m_{ij}]_{m \times n} \quad \text{where } m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Examples 1) $R = \{(2,1), (3,1), (3,2)\}$ where $A = \{1,2,3\}$ and $B = \{1, 2\}$.

Sol

$$M_R = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \rightarrow B \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \\ \downarrow \\ A & \end{array}$$

2) If $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3, b_4, b_5\}$ and

$$M_R = \begin{array}{c c} & \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

then $R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$

3) Represent the Relation $R = \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3)\}$.

on the set $\{1,2,3,4\}$ with a matrix

Sol

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

4) $R = \{(1,2), (2,1), (2,2), (3,3)\}$ on the set $\{1,2,3\}$

Sol

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

5) List the ordered pairs in the relation on $\{1,2,3\}$ corresponding to the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$R = \{(1,1), (2,2), (3,3)\}$

6) List the ordered pairs in the relation on $\{1,2,3,4\}$ corresponding to the matrix $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

Sol $R = \{(1,1), (1,2), (1,4), (2,1), (2,3), (3,2), (3,3), (3,4), (4,1), (4,3), (4,4)\}$

Determining the Properties of Relation from M_R which is a square matrix.

1) **Reflexive**:- If all the elements on the main diagonal of M_R are equal to 1, then R is Reflexive. Off diagonal elements can be 0 or 1.

Eg If $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

then R is Reflexive.

2) **Symmetric**:- If M_R is a symmetric matrix, then R is symmetric relation

Eg If $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, etc.

then R is Symmetric.

NOTE- Symmetric matrix means $A^T = A$.

Eg. $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ b_1 & b_2 & c_2 \\ c_1 & c_2 & c_3 \end{bmatrix}$; $A^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ b_1 & b_2 & c_2 \\ c_1 & c_2 & c_3 \end{bmatrix}$

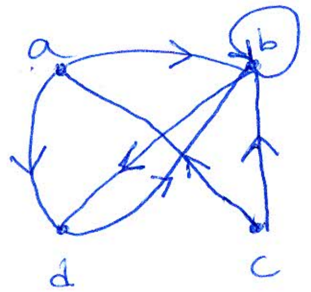
then A is symmetric

3) **Antisymmetric**:- If all the elements on the main diagonal of M_R are equal to 1 and off diagonal elements $m_{ij} = 1$ for $i \neq j$ and $m_{ji} = 0$.

Eg If $M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; $M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$.

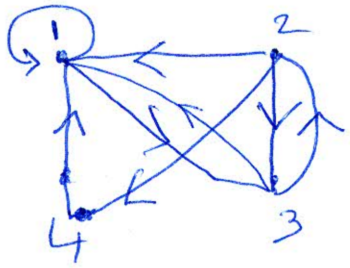
Representing Relations Using Digraphs

A directed graph with vertices a, b, c and d and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b) and (d, b) is represented as.



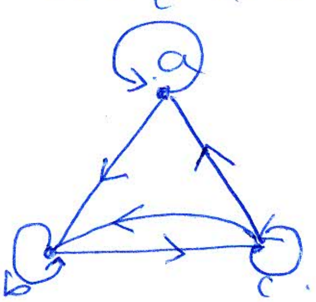
A relation R on a set A can be represented by the directed graph whose vertices are elements of A and ordered pairs $(a, b) \in R$ as edges.

Eg 1 $R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$ on set $A = \{1, 2, 3, 4\}$.

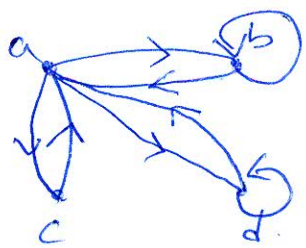


Eg 2

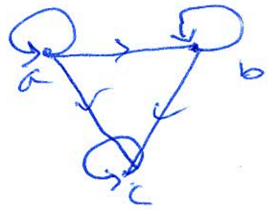
$R = \{(a,a), (a,b), (b,b), (b,c), (c,a), (c,b), (c,c)\}$ on $A = \{a, b, c\}$.



$R = \{(a,b), (a,c), (a,d), (b,a), (b,b), (c,a), (d,a), (d,d)\}$



Reflexive :- A Relation is Reflexive if and only if there is a loop at every vertex of the directed graph.



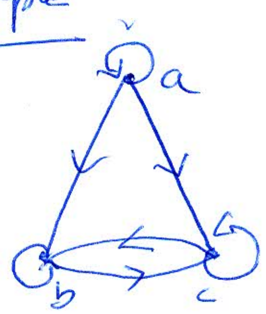
$R = \{(a,a), (a,b), (a,c), (b,b), (b,c), (c,c)\}$ on set $A = \{a, b, c\}$

Symmetric :- A Relation is Symmetric if and only if for every edge between distinct vertices in the digraph there is an edge in the opposite direction.

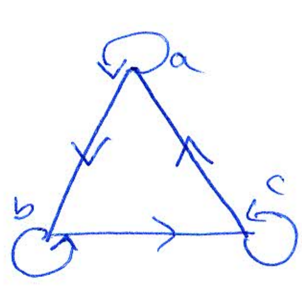
Antisymmetric :- A Relation is Antisymmetric if and only if there are never two edges in opposite directions between distinct vertices.

Transitive :- A Relation is transitive if and only if whenever there is an edge from vertex a to vertex b, and an edge from vertex b to vertex c, there is an edge from vertex a to vertex c.

Example



Reflexive
Not Symmetric
Not Antisymmetric.
Transitive.



Reflexive.
Not Symmetric
~~Not~~ Antisymmetric
Transitive

EQUIVALENCE RELATION: A relation R on a set A is said to be an equivalence relation if it is reflexive, symmetric and transitive.

Examples

1). If R is a relation on the set of real numbers such that aRb if and only if $a-b$ is a integer. Show that R is an equivalence relation

Solution: **Reflexive**

$a-a=0$ is an integer when a is a real number.

$\Rightarrow aRa$ (or) $(a,a)\in R, \forall a \in \mathbb{R}$

$\Rightarrow R$ is reflexive.

Symmetric

Let aRb (or) $(a,b)\in R$

Then $a-b$ is an integer.

$\Rightarrow a-b=K$. where K is integer.

$\Rightarrow b-a=-K$ where $-K$ is integer.

$\Rightarrow (b,a)\in R$ (or) bRa.

$\Rightarrow R$ is Symmetric.

Transitive Let aRb and bRc.

$\Rightarrow a-b$ is an integer ; $b-c$ is an integer

$\Rightarrow a-b=K_1$ and $b-c=K_2$

$\Rightarrow a-b+b-c=K_1+K_2$

$\Rightarrow a-c=(K_1+K_2)$ which is an integer

$\Rightarrow a R c.$

$\Rightarrow R$ is transitive.

Therefore R is an equivalence relation

2) Show that the relation $R = \{(a, b) \mid a \equiv b \pmod m\}$ is an equivalence relation on set of integers.

Sol. $a - a = 0$ is divisible by m

$\Rightarrow a \equiv a \pmod m$.

$\Rightarrow (a, a) \in R$

$\Rightarrow R$ is reflexive.

Let $a \equiv b \pmod m$

$\Rightarrow a-b$ is divisible by m

$\Rightarrow a-b = km$

$\Rightarrow b-a = (-k)m$

$\Rightarrow b-a$ is divisible by m

$\Rightarrow b \equiv a \pmod m$.

$\Rightarrow (b, a) \in R$ whenever $(a, b) \in R$

$\Rightarrow R$ is symmetric.

Let $a \equiv b \pmod m$ and $b \equiv c \pmod m$

$\Rightarrow a-b$ is divisible by m $\quad$ $b-c$ is divisible by m

$\Rightarrow a-b = k_1 m$ $\quad$ $b-c = k_2 m$

$\Rightarrow a-b+b-c = k_1 m + k_2 m$

$\Rightarrow a-c = (k_1+k_2) m$

$\Rightarrow a-c$ is divisible by m

$\Rightarrow a \equiv c \pmod m$

$\Rightarrow (a, c) \in R$ whenever $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow R$ is an equivalence relation.

3) Determine whether the relation represented by zero one matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is an equivalence relation.

Sol $M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

As all the main diagonal elements are 1

$\Rightarrow$ R is reflexive.

$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$; $M_R^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

As $M_R \neq M_R^T \Rightarrow R$ is not symmetric.

$\Rightarrow$ R is not equivalence relation.

4) $M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

All main diagonal elements are 1

$\Rightarrow$ R is reflexive.

$(M_R)^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

As $M_R = (M_R)^T \Rightarrow R$ is symmetric.

$R = \{(a,a), (a,c), (b,b), (b,d), (c,a), (c,c), (d,b), (d,d)\}$.

As $(a,a) \in R, (a,c) \in R \Rightarrow (a,c) \in R$
$(a,c) \in R, (c,a) \in R \Rightarrow (a,a) \in R$
$(a,c) \in R, (c,c) \in R \Rightarrow (a,c) \in R$
$(b,b) \in R, (b,d) \in R \Rightarrow (b,d) \in R$
$(b,d) \in R, (d,b) \in R \Rightarrow (b,b) \in R$
$(b,d) \in R, (d,d) \in R \Rightarrow (b,d) \in R$

$(c,a) \in R, (a,a) \in R \Rightarrow (c,a) \in R$
$(c,a) \in R, (a,c) \in R \Rightarrow (c,c) \in R$
$(c,c) \in R, (c,a) \in R \Rightarrow (c,a) \in R$

$(d,b) \in R, (b,b) \in R \Rightarrow (d,b) \in R$

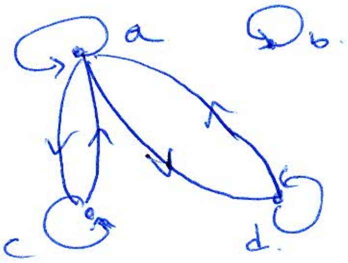
$(d,b) \in R, (b,d) \in R \Rightarrow (d,d) \in R$

$(d,d) \in R, (d,b) \in R \Rightarrow (d,b) \in R$

$\Rightarrow$ R is transitive

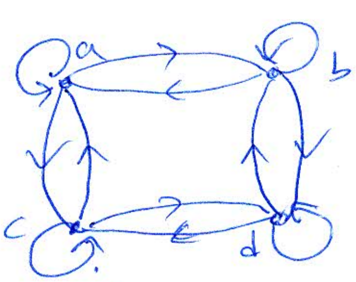
$\Rightarrow$ R is an equivalence relation.

5) Determine whether the relation with the directed graph is an equivalence relation.



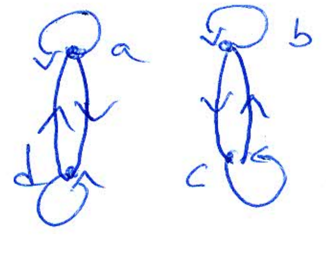
R is reflexive as there is a loop at every vertex.
R is symmetric as there is an edge between a and c followed by c and a; a and d followed by d and a.
R is not transitive $(c,a) \in R, (a,d) \in R \Rightarrow (c,d) \notin R$
~~$(a,d) \in R, (d,a) \in R \Rightarrow (a,a) \in R$~~
$\Rightarrow$ R is not an equivalence relation.

6)



R is reflexive
R is symmetric
R is not transitive
since $(a,b) \in R, (b,d) \in R$
but $(a,d) \notin R$
R is not equivalence relation

7)



R is reflexive
R is symmetric
R is transitive
R is an equivalence relation.

PARTIAL ORDERING (OR) PARTIAL ORDER

A relation R on a set S is called a Partial ordering or Partial order if it is reflexive, antisymmetric and transitive.

POSET:- A set S together with a partial ordering R is called a Partially ordered set or Poset and is denoted by (S, R).

Examples

1) $(Z, \geq)$. 2) $(Z^+, |)$ divisibility.

Problem:- Show that the relation $R = \{(a,b)$ such that a divides b is not an equivalence relation on set of Positive integers.

Solution:-

a divides a.

$\Rightarrow (a, a) \in R$

$\Rightarrow$ R is reflexive.

Let a divides b. i.e., $(a,b) \in R$

$\Rightarrow b = aq$ (the remainder when b is divided by a is zero).

$\Rightarrow a = \left(\frac{1}{q}\right) b$.

$\Rightarrow$ b does not divide a. i.e., $(b, a) \notin R$

$\Rightarrow$ R is not symmetric. (For example 3 divides 18 but 18 does not divide 3).

Let a divides b i.e., $(a, b) \in R$

$\Rightarrow b = a q_1$

Let b divides c i.e., $(b, c) \in R$

$\Rightarrow c = b q_2$

$\Rightarrow c = a(q_1 q_2)$

$\Rightarrow$ a divides c i.e., $(a, c) \in R$

$\Rightarrow$ R is transitive.

R is not an equivalence relation