COUNTING

CHAPTER-6

WEEK-10

THE PRODUCT RULE:- If there are n_1 ways to do the first task and n_2 ways to do the second task, then the entire task can be done in $n_1 n_2$ ways.

Examples ① How many bit strings of length eight are there.

Sol:- $\underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2} = 2^8$

② An office building contains 27 floors and has 37 offices on each floor. How many offices are there in the building.

Sol $\underline{35} \cdot \underline{27} = 35 \times 27 = 945$

③ How many Different License Plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (No sequences of letters are prohibited)

Sol. $\underbrace{\underline{26}\ \underline{26}\ \underline{26}}_{\text{upper case letters}}\ \underbrace{\underline{10}\ \underline{10}\ \underline{10}}_{\text{Digits}} = 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$
$= 17,576,000$

④ Bit strings of length ten both begin and end with 1

Sol $\underline{1}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{1} = 2^8$

*(5) Number of functions from $A \to B$ where set A has m elements, set B has n elements. $\underline{n^m}$

*(6) Number of one-one functions from $A \to B$ where set A has m elements, set B has n elements and $m \leq n$ are $\underline{n(n-1)(n-2) \cdots (n-m+1)}$

Examples 1) Number of functions from set with 3 elements to a set with 4 elements $\underline{4^3} = 64$

2) Number of one-one functions from set with 3 elements to a set with 4 elements: $\underline{4(4-1)(4-2) = 4 \cdot 3 \cdot 2 = 24}$

THE SUM RULE:- If a task can be done either in one of n_1 ways (or) in one of n_2 ways, then the task can be done in $(n_1 + n_2)$ ways. Here the n_1 ways and n_2 ways are disjoint.

Example:- There are 18 mathematics and 325 computer science majors at a college. In how many ways can one representive be picked who is either a mathematics major or a computer science major.

Sol $18 + 35 = 53$

Example:- A computer system password has six to eight characters long, where each character is an uppercase letter or a digit. How many passwords are there, if each password must contain atleast one digit.

Sol. Passwords of length 6: $\underbrace{(36\ 36\ 36\ 36\ 36\ 36)}_{\text{Passwords with letters and digits}} - \underbrace{(26\ 26\ 26\ 26\ 26\ 26)}_{\text{Passwords with letters}} = 36^6 - 26^6$

~~Passwords of length 7:~~

~~Passwords of length 8.~~

Similarly Passwords with length 7 $= 36^7 - 26^7$

Passwords with length 8 $= 36^8 - 26^8$

<u>The Possible Passwords</u> $36^6 - 26^6 + 36^7 - 26^7 + 36^8 - 26^8$

$= 2684483063360$

<u>THE SUBTRACTION RULE</u>:- If a task can be done in either n_1 ways or n_2 ways, then the numbers of ways to do the task is.

$(n_1 + n_2)$ – the number of ways common to n_1 and n_2.

<u>Example</u>

1) $|A \cup B| = |A| + |B| - |A \cap B|$

2) How many bit strings of length eight either start with 1 bit or end with the two bits 00?

<u>Sol</u>

$$1\,\underbrace{\_\,\_\,\_\,\_\,\_\,\_\,\_}_{2^7} + \underbrace{\_\,\_\,\_\,\_\,\_\,\_}_{2^6}\,0\,0 - 1\,\underbrace{\_\,\_\,\_\,\_\,\_}_{2^5}\,0\,0$$

$= 2^7 + 2^6 - 2^5 = 160$

<u>THE PIGEONHOLE PRINCIPLE</u>:- If $(k+1)$ or more objects are placed in k boxes, then there is atleast one box containing two or more objects.

Examples

1. Among a group of 367 people, there will be atleast two with the same birthday.
   (Reason: There are only 366 possible birthdays)

2. How many students must be there in the class so that atleast two students receive the same score, if the exam is graded on a scale from 0 to 100 points

Ans:- 102 students. (0 to 100 means 101 possible scores).

GENERALIZED PIGEONHOLE PRINCIPLE:-

If N objects are placed into K boxes, then there will be atleast one box containing atleast $\left\lceil \frac{N}{K} \right\rceil$ objects.

Example

Among 100 people, how many are born in the same month.

Sol:- $N = 100$, $K = 12$

$$\left\lceil \frac{N}{K} \right\rceil = \left\lceil \frac{100}{12} \right\rceil = \lceil 8.333 \rceil = 9$$

FACTORIAL: The product of first n natural numbers is called as $n!$ (read as n factorial)

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-2)(n-1)n$$

(or) $n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$

Eg $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$ (or) $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Eg $7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$ (or) $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$= 7 \cdot 6 \cdot 5 \cdot 4!$
$= 7 \cdot 6 \cdot 5!$
$= 7 \cdot 6!$

* NOTE:- $0! = 1$

PERMUTATIONS :- The number of ways to arrange a specified number of distinct elements from a set of a particular size, where the order of these elements matters.

1) The number of r - permutations of a set with n distinct elements is denoted by $P(n,r)$ and is given by

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

(or)

$${}^{n}P_{r} = P(n,r) = \frac{n!}{(n-r)!} \quad \text{where } 1 \le r \le n$$

when repetition of elements are not allowed.

2) The number of r - permutations of a set with n distinct elements, when repetition of elements are allowed is

$$P(n,r) = n \cdot n \cdot n \cdots n \quad (r\text{-times})$$

$$= n^{r}$$

NOTE:- The number of r - permutations of a set with n distinct elements, is nothing but filling r - blank spaces using n - objects (or) elements.

$P(n,r) =$

n	$(n-1)$	$(n-2)$	. . . .	$n-r+1$
FIRST BLANK SPACE	Second blank Space	Third blank Space		r^{th} blank Space

Examples

1) Find 5_{P_2} (or) $P(5,2)$

Sol: $P(5,2) = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5\cdot4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 20$

2) Find $P(10,3)$

Sol: $P(10,3) = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10\cdot8\cdot9\cdot\cancel{7!}}{\cancel{7!}} = 720$

3) Find $P(n,0)$ (or) n_{P_0} (And) $P(n,n)$ or ^nP_n

Sol: $P(n,0) = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$ | $P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$

4) In how many ways can we arrange three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

Sol
i) $P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\cdot4\cdot3\cdot\cancel{2}\cdot1}{\cancel{2}\cdot1} = 60$

ii) $P(5,5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{1} = 120$

5) If $S = \{a,b,c,d,e\}$. Find 2-permutations, 3-permutations, 4-permutation and 5-permutation of S.

Sol $S = \{a,b,c,d,e\} \Rightarrow n=5$

2-permutations $\Rightarrow r=2 \Rightarrow P(5,2) = \frac{5!}{3!} = 20$

3-permutations $\Rightarrow r=3 \Rightarrow P(5,3) = \frac{5!}{2!} = 60$

4-permutation $\Rightarrow r=4 \Rightarrow P(5,4) = \frac{5!}{1!} = 120$

5-permutation $\Rightarrow r=5 \Rightarrow P(5,5) = \frac{5!}{0!} = 120$

6) How many permutations of the letters ABCDEFGH contain the string ABC.

Sol. Here every permutation must contain ABC. So tie ABC and treat them as one block. Wherever this block goes ABC goes along with.

Now

(ABC) D E F G H

$\Rightarrow n = 6$

Now $P(n,n) = P(6,6) = \frac{6!}{0!} = 720$

7) How many strings of length 3 can be formed from uppercase letters of English, when repetition is allowed

Sol $n = 26,\ r = 3$ and repetition is allowed

Required Permutations $= n^r = 26^3$

PERMUTATIONS WITH INDISTINGUISHABLE OBJECTS:

(same)

The number of permutations of n objects where
n_1 are indistinguishable objects (same objects) of type 1
n_2 " " " " " of type 2
n_3 " " " " of type 3
so on.
n_k " " " " of type k

is

$$\frac{n!}{n_1!\, n_2!\, n_3! \ldots n_k!}$$

Examples: Find the number of Permutations of the letters of the following words.

(i) SUCCESS (ii) MISSISSIPPI (iii) ABRACADABRA

Sol

SUCCESS $\Rightarrow n = 7$

number of S's = 3, number of U's = 1
number of C's = 2, number of E's = 1

Required Permutations = $\dfrac{7!}{3!\,2!\,1!\,1!} = \dfrac{7\cdot 6\cdot 5\cdot 4\cdot \cancel{3!}}{\cancel{3!}\; 2\cdot 1} = 420$

MISSISSIPPI $\Rightarrow n = 11$

S's = 4, M's = 1
I's = 4, P's = 2

Required Permutations = $\dfrac{11!}{4!\,4!\,1!\,2!}$

ABRACADABRA $\Rightarrow n = 11$

A's = 5, C's = 1, R's = 2
B's = 2, D's = 1

Required Permutations = $\dfrac{11!}{5!\,2!\,1!\,1!\,2!}$

COMBINATIONS: Refers to number of ways to select a particular number of elements from a particular set of size n, where the order of elements selected does not matter.

The number of r-combinations of a set with n elements is denoted by $C(n, r)$ (or) $^{n}C_{r}$ (or) $\binom{n}{r}$ and is given by $C(n,r) = \dfrac{n!}{r!\,(n-r)!}$ where $0 \le r \le n$.

(or) $C(n,r) = \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$

When repetition is not allowed.

NOTE

$$^{n}c_{r} = \frac{^{n}p_{r}}{r!} \quad (or) \quad c(n,r) = \frac{P(n,r)}{r!}$$

Examples

1) Find 5_{c_2} (or) $c(5,2)$

Sol $5_{c_2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!\,3!} = \frac{5\cdot 4\cdot \cancel{3!}}{2\cdot 1\cdot \cancel{3!}} = 10$

2) Find 6_{c_3} (or) $c(6,3)$

Sol $6_{c_3} = \frac{6!}{3!(6-3)!} = \frac{6!}{3!\,3!} = \frac{6\cdot 5\cdot 4\cdot \cancel{3!}}{\cancel{3!}\;3\cdot 2\cdot 1} = 20$

3) Find $c(n,n)$ (or) n_{c_n}

Sol $n_{c_n} = \frac{n!}{n!(n-n)!} = \frac{\cancel{n!}}{\cancel{n!}\,0!} = \frac{1}{1} = 1$

4) Find $c(3,3)$ (or) 3_{c_3}

Sol $c(3,3) = \frac{3!}{3!(3-3)!} = \frac{\cancel{3!}}{\cancel{3!}\,0!} = \frac{1}{1} = 1$

5) Find 7_{c_1} or $c(7,1)$

Sol $7_{c_1} = \frac{7!}{1!(7-1)!} = \frac{7!}{1!\,6!} = \frac{7\cdot \cancel{6!}}{\cancel{6!}} = 7$

6) Show that $C(n,r) = \dfrac{P(n,r)}{r!}$

Sol. $C(n,r) = \dfrac{n!}{r!\,(n-r)!}$ — (1)

$P(n,r) = \dfrac{n!}{(n-r)!}$ — (2)

$$\frac{C(n,r)}{P(n,r)} = \frac{\dfrac{\cancel{n!}}{r!\,\cancel{(n-r)!}}}{\dfrac{\cancel{n!}}{\cancel{(n-r)!}}}$$

$$\frac{C(n,r)}{P(n,r)} = \frac{1}{r!} \Rightarrow C(n,r) = \frac{P(n,r)}{r!}$$

7) Show that $C(n,r) = C(n,n-r)$

Sol $C(n,r) = \dfrac{n!}{r!\,(n-r)!}$ — (1)

$$C(n,n-r) = \frac{n!}{(n-r)!\,(n-(n-r))!}$$

$$= \frac{n!}{(n-r)!\,r!} \quad \text{— (2)}$$

From (1) & (2)

$C(n,r) = C(n,n-r)$

8. PASCAL'S TRIANGLE (IDENTITY)

Show that $c(n, r-1) + c(n, r) = c(n+1, r)$

(or)

$$^{n}c_{r-1} + {}^{n}c_{r} = {}^{(n+1)}c_{r}$$

Sol Consider LHS

$$c(n, r-1) + c(n,r) = \frac{n!}{(r-1)!(n-(r-1))!} + \frac{n!}{r!(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!}$$

$$= \frac{r\,n!}{r(r-1)!(n-r+1)!} + \frac{n!(n-r+1)}{r!(n-r+1)(n-r)!}$$

$$= \frac{r\,n!}{r!(n-r+1)!} + \frac{n!(n-r+1)}{r!(n-r+1)!}$$

$$= \frac{r\,n! + n!(n-r+1)}{r!(n-r+1)!}$$

$$= \frac{n!(\cancel{r} + n - \cancel{r} + 1)}{r!(n-r+1)!}$$

$$= \frac{(n+1)\,n!}{r!(n-r+1)!}$$

$$= \frac{(n+1)!}{r!(n-r+1)!}$$

$= c(n+1, r)$

$=$ RHS

Examples on Pascals triangle

$$^1c_0 + {^1c_1} = {^{(1+1)}c_1} = {^2c_1}$$

$$^2c_0 + {^2c_1} = {^{(2+1)}c_1} = {^3c_1}$$

$$^2c_1 + {^2c_2} = {^{(2+1)}c_2} = {^3c_2}$$

$$^3c_0 + {^3c_1} = {^{(3+1)}c_1} = {^4c_1}$$

$$^3c_1 + {^3c_2} = {^{(3+1)}c_2} = {^4c_2}$$

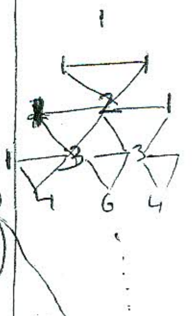
Means

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$



9) How many ways can we select three students from a group of 5 students.

Sol $C(5,3) = \dfrac{5!}{3!\,(5-3)!} = \dfrac{5!}{3!\,2!} = \dfrac{5\cdot 4\cdot \cancel{3!}}{\cancel{3!}\,2\cdot 1} = 10$

10) How many ways are there to select 5 players from a 10 member tennis team

Sol $C(10,5) = \dfrac{10!}{5!\,(10-5)!} = \dfrac{10!}{5!\,5!} = \dfrac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot \cancel{5!}}{\cancel{5!}\,5\cdot 4\cdot 3\cdot 2\cdot 1}$

$= 252$

11) There are 11 mathematics faculty and 9 computer science faculty in an university. How many ways a committee of 7 members can be formed so as to include 4 math faculty and 3 computer science faculty

Sol: $C(11,4)\cdot C(9,3) = \dfrac{11!}{4!\,7!}\;\dfrac{9!}{3!\,6!}$

BINOMIAL THEOREM:- If x and y are variables and n is a positive integer, then

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n$$

(or)

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

Examples

1) Write the expansion of $(x+y)^5$

Sol

$$(x+y)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4y^1 + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5$$

$$= \frac{5!}{0!(5-0)!}x^5 + \frac{5!}{1!(5-1)!}x^4y^1 + \frac{5!}{2!(5-2)!}x^3y^2 + \frac{5!}{3!(5-3)!}x^2y^3 + \frac{5!}{4!(5-4)!}xy^4 + \frac{5!}{5!(5-5)!}y^5$$

$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

2) What is the coefficient of x^8y^7 in the expansion of $(x+y)^{15}$

Sol.-

$$(x+y)^{15} = \sum_{r=0}^{15} \binom{15}{r} x^{15-r} y^r$$

To get coefficient of x^8y^7, Put $r = 7$ $\Rightarrow \binom{15}{7}x^{15-7}y^7 \Rightarrow \binom{15}{7}x^8y^7$

Therefore, coefficient of $x^8y^7 = \binom{15}{7}$

$$= \frac{15!}{7!(15-7)!} = \frac{15!}{7!\,8!} =$$

3) Find the coefficient of $x^7 y^5$ in the expansion of $(2x-3y)^{12}$

Sol $(2x-3y)^{12} = [2x+(-3y)]^{12} = \sum_{r=0}^{12} \binom{12}{r} (2x)^{12-r} (-3y)^{r}$

To get coefficient of $x^7 y^5$ put $r=5$

$\Rightarrow \binom{12}{5} (2x)^{12-5} (-3y)^5$

$\Rightarrow \binom{12}{5} (2x)^7 (-3y)^5$

Coeffecient of $x^7 y^5 = \binom{12}{5} 2^7 (-3)^5$

4) Find the coefficient of $x^{12} y^7$ in the expansion of $(3x+4y)^{19}$

Sol $(3x+4y)^{19} = \sum_{r=0}^{19} \binom{19}{r} (3x)^{19-r} (4y)^{r}$

To get coefficient of $x^{12} y^7$, Put $r=7$

$\Rightarrow \binom{19}{7} (3x)^{19-7} (4y)^7$

$\Rightarrow \binom{19}{7} (3x)^{12} (4y)^7$

$\Rightarrow \binom{19}{7} 3^{12} 4^7 x^{12} y^7$

Coefficient of $x^{12} y^7 = \binom{19}{7} 3^{12} 4^7$

$= {}^{19}C_7 \, 3^{12} \, 4^7$

NOTE

1) $^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$

(or)

$$\sum_{K=0}^{n} {}^nC_K = 2^n \quad \text{(or)} \quad \sum_{K=0}^{n} \binom{n}{K} = 2^n$$

2) $^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \cdots + (-1)^n\, {}^nC_n = 0$

(or)

$$\sum_{K=0}^{n} (-1)^K\, {}^nC_K = 0 \quad \text{(or)} \quad \sum_{K=0}^{n} (-1)^K \binom{n}{K} = 0$$

3) $^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots$

COMBINATIONS WITH REPETITION:-

The r- combinations from a set with n elements when repetition of elements is allowed are

$C(n+r-1, r)$ (or) $C(n+r-1, n-1)$

Examples 1) How many ways are there to select five bills from a cash box containing 1\$, 2\$, 5\$, 10\$, 20\$, 50\$ and 100\$ bills and there are atleast five bills of each type.

Sol $C(7+5-1, 5) = C(11, 5)$

$= \frac{11!}{5!\,6!} = 462.$

(2) How many solutions does the equation $x + y + z = 11$ have where x, y, z are non negative integers

Sol ~~C(3+11)~~ Here $n = 3$, $r = 11$

$$C(3+11-1, 11) = C(13, 11) = \frac{13!}{11!\,2!} = 78$$

(3) A cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen.

Sol $n = 4$, $r = 6$.

$$C(4+6-1, 6) = C(9, 6) = \frac{9!}{6!\,3!} = 84.$$

DISTRIBUTING OBJECTS INTO BOXES

OBJECTS	BOXES	Number of ways
DISTINGUISHABLE objects n	Distinguishable boxes K.	$\frac{n!}{n_1!\,n_2!\ldots n_K!}$
Indistinguishable objects n	Distinguishable boxes K	$C(K+n-1, K)$
Distinguishable objects n	Indistinguishable boxes K	
Indistinguishable objects n	Indistinguishable boxes K	

Example for Distinguishable objects and Indistinguishable boxes :-

How many ways are there to put 4 different employees into 3 indistinguishable offices, when each office can contain any number of employees.

Sol

All 4 in one office – {A, B, C, D} —— 1

Three in one office and fourth in different office.

{{A, B, C}, {D}}, {{A, B, D}, {C}}

{{A, C, D}, {B}}, {{B, C, D}, {A}} —— 4

Two in one office and two in second office.

{{A, B}, {C, D}}, {{A, C}, {B, D}}, {{A, D}, {B, C}} – 3

Two in one office and one each into each of the remaining two offices.

{{A, B}, {C}, {D}}, {{A, C}, {B}, {D}}, {{A, D}, {B}, {C}}

{{B, C}, {A}, {D}}, {{B, D}, {A}, {C}}, {{C, D}, {A}, {B}}

—— 6

14 ways

Example for Indistinguishable objects and Indistinguishable boxes :-

How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books

Sol · One box is used – All 6 books in one box – 1 way

Two boxes are used – 5, 1 ; 4, 2 ; 3, 3 } – 3 ways

Three boxes are used – 4, 1, 1 ; 3, 2, 1 ; 2, 2, 2 } – 3 ways

Four boxes are used – 3, 1, 1, 1 ; 2, 2, 1, 1. } – 2 ways

9 ways

ADVANCED COUNTING TECHNIQUES

CHAPTER-8

WEEK-11

LINEAR HOMOGENEOUS RECURRENCE RELATION

A recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_K a_{n-K}$$

where $c_K \neq 0$ is called a linear homogeneous recurrence relation of degree K with constant coefficients.

Examples:

1) $P_n = (1.11) P_{n-1}$ is linear homogeneous of degree 1
2) $f_n = f_{n-1} + f_{n-2}$ is linear homogeneous of degree 2
3) $a_n = a_{n-5}$ linear homogeneous of degree 5
4) $a_n = a_{n-1} + a_{n-2}^2$ is not linear.
5) $H_n = 2H_{n-1} + 1$ is not homogeneous
6) $B_n = n B_{n-1}$ is not a homogeneous recurrence relation with constant coefficients.
7) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$ – linear – Degree 3.
8) $a_n = 2n\, a_{n-1} + a_{n-2}$ not homogeneous with constant coefficients.
9) $a_n = a_{n-1} + n$ not linear.
10) $a_n = a_{n-1}^2$ not linear.

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients.

Step 1 Write the characteristic eqn. of the recurrence relation by putting $a_n = r^n$ and solve the characteristic equation to get roots

Step 2 If the roots of the characteristic equation are real and distinct (say $r_1, r_2, \ldots r_k$) then, the solution is

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r^k.$$

Step 3 If the roots of the characteristic equation are real and identical (say $r_1, r_1, r_3, r_4, \ldots r_k$) then, the solution is

$$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n + \alpha_3 r_3^n + \alpha_4 r_4^n + \cdots + \alpha_k r_k^n$$

Step 4 If the roots are $r_1, r_1, r_1, r_4, r_5, \ldots r_k$. Then solution is

$$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n + \alpha_3 n^2 r_1^n + \alpha_4 r_4^n + \cdots + \alpha_k r_k^n$$

EXAMPLES.

1) Solve the recurrence relations.

a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$.

Sol: $a_n = 2 \cdot a_{n-1}$ —— (1)

Degree $= 1$

Put $a_n = r^n$ in (1)

$\Rightarrow r^n = 2r^{n-1}$

$\Rightarrow r = 2$

Solution of 1 is

$$a_n = c_1 (2)^n \quad \text{--- (2)}$$

Given $a_0 = 3 \Rightarrow$ Put $n=0$ in (2)

$\Rightarrow c_1 (2)^0 = 3$

$\Rightarrow c_1 = 3$.

Solution of (1) is $a_n = 3(2)^n$.

b) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1, a_1 = 2$

Sol $a_n = 5a_{n-1} - 6a_{n-2}$ --- (1)

Degree $= 2$

Put $a_n = r^n$ in (1)

$r^n = 5r^{n-1} - 6r^{n-2}$

$\Rightarrow r^2 = 5r - 6$

$\Rightarrow r^2 - 5r + 6 = 0$

$\Rightarrow (r-3)(r-2) = 0$

$\Rightarrow r = 2, 3$

Solution of (1) is $a_n = c_1 (2)^n + c_2 (3)^n$ --- (2)

Given $a_0 = 1 \Rightarrow$ Put $n=0$ in (2)

$\Rightarrow c_1 (2)^0 + c_2 (3)^0 = 1$

$\Rightarrow c_1 + c_2 = 1$ --- (3)

Given $a_1 = 2 \Rightarrow$ Put $n=0$ in (2)

$\Rightarrow 2 = c_1 (2)^1 + c_2 (3)^1$

$\Rightarrow 2 = 2c_1 + 3c_2$ --- (4)

Solving (3) & (4)

$c_1 + c_2 = 1$

$2c_1 + 3c_2 = 2$

$\Rightarrow c_2 = 0, c_1 = 1$

Solution of (1) is $a_n = (2)^n$

c) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$

Sol $a_n = -4a_{n-1} - 4a_{n-2}$ — (1)

Degree = 2

Put $a_n = r^n$ in (1)

$\Rightarrow r^n = -4r^{n-1} - 4r^{n-2}$

$\Rightarrow r^2 = -4r - 4$

$\Rightarrow r^2 + 4r + 4 = 0$

$\Rightarrow (r+2)^2 = 0$

$\Rightarrow r = -2, -2$

Solution of (1) is

$a_n = c_1(-2)^n + c_2 n(-2)^n$ — (2)

Given $a_0 = 0 \Rightarrow$ Put $n = 0$ in (2)

$\Rightarrow c_1(-2)^0 + c_2\, 0\, (-2)^0 = 0$

$\Rightarrow c_1 = 0$

Given $a_1 = 1 \Rightarrow$ put $n = 1$ in (2)

$\Rightarrow c_1(-2)^1 + c_2\, 1\, (-2)^1 = 1$

$\Rightarrow c_1(-2) + c_2(-2) = 1$

$\Rightarrow 0 - 2c_2 = 1 \Rightarrow c_2 = -1/2$

Solution of (1)

$a_n = 0(-2)^n + (-\frac{1}{2})n(-2)^n$

$= -\frac{n}{2}(-2)^n$

d) $a_n = 4a_{n-1} - 4a_{n-2}$, $n \geq 2$, $a_0 = 6$, $a_1 = 8$

Sol $a_n = 4a_{n-1} - 4a_{n-2}$ — ①

Degree $= 2$

Put $a_n = r^n$ in ①

$r^n = 4r^{n-1} - 4r^{n-2}$

$\Rightarrow r^2 = 4r - 4$

$\Rightarrow r^2 - 4r + 4 = 0$

$\Rightarrow (r-2)^2 = 0$

$\Rightarrow r = 2, 2$

Solution of ① is

$a_n = c_1(2)^n + c_2 n(2)^n$ — ②

Given $a_0 = 6 \Rightarrow$ Put $n = 0$ in ②

$c_1(2)^0 + c_2 0(2)^0 = 6$

$\Rightarrow c_1 = 6$

Given $a_1 = 8 \Rightarrow$ Put $n = 1$ in ②

$c_1(2)^1 + c_2 1(2)^1 = 8$

$\Rightarrow 2c_1 + 2c_2 = 8$

$\Rightarrow c_1 + c_2 = 4$

$\Rightarrow 6 + c_2 = 4 \Rightarrow c_2 = -2$

Solution of ① is

$a_n = 6(2)^n - 2n(2)^n$

$\Rightarrow a_n = (6-2n)2^n$

e) Solve $f_n = f_{n-1} + f_{n-2}$; $f_0 = 0$, $f_1 = 1$

Sol

$$f_n = f_{n-1} + f_{n-2} \quad \text{--- } 1$$

Degree $= 2$

Put $f_n = r^n$ in 1

$\Rightarrow r^n = r^{n-1} + r^{n-2}$

$\Rightarrow r^2 = r + 1$

$\Rightarrow r^2 - r - 1 = 0$

$\Rightarrow r = \dfrac{1 \pm \sqrt{1+4}}{2}$

$= \dfrac{1 \pm \sqrt{5}}{2}$

$\Rightarrow r = \dfrac{1+\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}$.

Solution of 1 is

$$f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{--- } 2$$

Given $f_0 = 0 \Rightarrow$ Put $n=0$ in 2

$c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = 0$

$\Rightarrow c_1 + c_2 = 0$ --- 3

Given $f_1 = 1 \Rightarrow$ Put $n=1$ in 2

$c_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$

$\Rightarrow c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$ --- 4

Solving 3 & 4

$c_1 + c_2 = 0$

$\Rightarrow c_1 = -c_2$.

But: $c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$\Rightarrow -c_2\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$\Rightarrow \frac{c_2}{2}\left(\cancel{-1} - \sqrt{5} + \cancel{1} - \sqrt{5}\right) = 1$

$\Rightarrow c_2 = -1/\sqrt{5}$ and $c_1 = 1/\sqrt{5}$

Solution of 1 is $f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$.

f) Solve. $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$

Sol

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} \quad \text{— (1)}$$

Degree $= 3$

Put $a_n = r^n$ in (1)

$$r^n = 6r^{n-1} - 11r^{n-2} + 6r^{n-3}$$

$$\Rightarrow r^3 = 6r^2 - 11r + 6.$$

$$\Rightarrow r^3 - 6r^2 + 11r - 6 = 0$$

$$\Rightarrow (r-1)(r-2)(r-3) = 0$$

$$\Rightarrow r = 1, 2, 3.$$

1	1	-6	11	-6
	0	1	-5	6
2	1	-5	6	0
	0	2	-6	
	1	-3	0	

Solution of (1) is $a_n = c_1(1)^n + c_2(2)^n + c_3(3)^n$

g) Solve. $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$. — (1)

Sol. Degree $= 3$

Put $a_n = r^n$ in (1)

$$\Rightarrow r^n = -3r^{n-1} - 3r^{n-2} - r^{n-3}$$

$$\Rightarrow r^3 = -3r^2 - 3r - 1$$

$$\Rightarrow r^3 + 3r^2 + 3r + 1 = 0$$

$$\Rightarrow (r+1)^3 = 0$$

$$\Rightarrow r = -1, -1, -1$$

Solution of (1) is

$$a_n = c_1(-1)^n + c_2 n(-1)^n + c_3 n^2 (-1)^n$$

GENERATING FUNCTIONS:-

The generating function for the sequence of real numbers $a_0, a_1, \ldots a_k, \ldots$ is the infinite series $a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots$

(or)

$$\sum_{k=0}^{\infty} a_k x^k$$

Eg:- 1) The G.F for the sequence $\{a_k\}$ with $a_k = 3$

is $\sum_{k=0}^{\infty} 3x^k = 3 + 3x + 3x^2 + \cdots$

2) The G.F for the sequence $\{a_k\}$ with $a_k = 2^k$

is $\sum_{k=0}^{\infty} 2^k x^k = 1 + 2x + 2^2 x^2 + \cdots$

Generating function for finite sequence:

The generating function for the finite sequence of real numbers $a_0, a_1, a_2, \ldots, a_n$ is $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

(or)

$$\sum_{k=0}^{n} a_k x^k$$

Eg: 1) The G.F for the sequence 1, 1, 1, 1, 1, 1

is $1 + x + x^2 + x^3 + x^4 + x^5$

(or)

$$\frac{x^6 - 1}{x - 1}.$$

NOTE-

$$\frac{1}{1-ax} = (1-ax)^{-1} = 1 + ax + a^2x^2 + \cdots$$

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + \cdots$$

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + \cdots$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

EXTENDED BINOMIAL COEFFICIENT

If n is a real number and r is a positive integer, then the extended binomial coefficient is denoted by $\binom{n}{r}$ and is defined as

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

Eg

$$\binom{-2}{3} = \frac{(-2)(-2-1)(-2-2)}{3!} = \frac{(-2)(-3)(-4)}{3\cdot 2\cdot 1} = -4$$

$$\binom{1/2}{3} = \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\cdot 2\cdot 1} = \frac{1}{16}.$$

$$\binom{-3}{5} = \frac{(-3)(-3-1)(-3-2)(-3-3)(-3-4)}{5!} = \frac{(-3)(-4)(-5)(-6)(-7)}{5\cdot 4\cdot 3\cdot 2\cdot 1} = -21$$

$$\binom{-1/3}{5} = \frac{(-\frac{1}{3})(-\frac{1}{3}-1)(-\frac{1}{3}-2)(-\frac{1}{3}-3)(-\frac{1}{3}-4)}{5!} = \frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})(-\frac{10}{3})(-\frac{13}{3})}{5\cdot 4\cdot 3\cdot 2\cdot 1} = -\frac{91}{36}$$

Examples:- 1) Find a closed form for the generating function formed by 2, 4, 8, 16, 32, 64, ...

Sol

$$G.F = 2 + 4x + 8x^2 + 16x^3 + 32x^4 + 64x^5 + \cdots$$

$$= 2(1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots)$$

$$= 2(1-2x)^{-1}$$

$$= \frac{2}{1-2x}$$

2) Find a closed form for the generating function for the sequence. −3, 3, −3, 3, −3, 3, ...

Sol

$$G.F = -3 + 3x - 3x^2 + 3x^3 - 3x^4 + 3x^5 \cdots$$

$$= -3(1 - x + x^2 - x^3 + x^4 - x^5 + \cdots)$$

$$= -3(1+x)^{-1}$$

$$= \frac{-3}{1+x}.$$

3) Find a generating function for the sequence 0, 0, 0, 1, 2, 3, 4, 5, ...

Sol

$$G.F = 0 + 0x + 0x^2 + 1x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + \cdots$$

$$= x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + \cdots$$

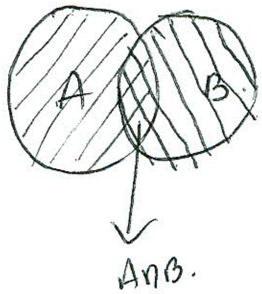
$$= x^3(1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots)$$

$$= x^3(1-x)^{-2}$$

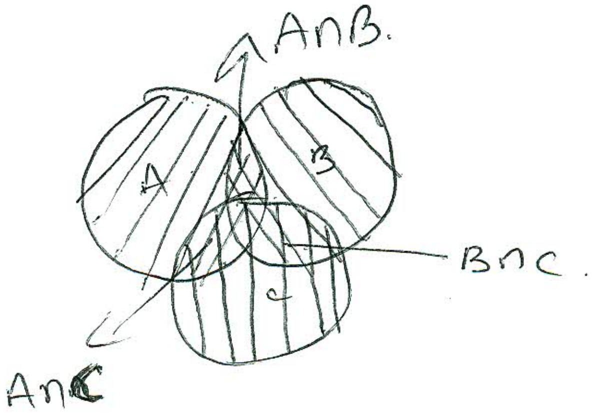
$$= \frac{x^3}{(1-x)^2}$$

THE PRINCIPLE OF INCLUSION - EXCLUSION

$$|A \cup B| = |A| + |B| - |A \cap B|$$



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$



Examples In a discrete mathematics class every student is a major in computer science or mathematics. The number of students having CS as a major is 32; the number of students having mathematics as a major is 23 and in both is 9. How many students are there in the class

Sol

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= |32| + |23| - |9|$$
$$= 46.$$

(2) There are 2504 CS students at a school. Of these, 1876 have taken Java, 999 have taken Linux, 345 have taken 'C'. Further, 876 have taken courses in both Java and Linux, 231 have taken courses in both Linux and C, and 290 have taken courses in both Java and C. If 189 of these students have taken courses in Linux, Java and C, how many of these 2504 students have not taken a course in any of these three programming languages.

Sol

Total Students = 2504

$|J| = 1876$

$|L| = 999$

$|C| = 345$

$|J \cap L| = 876$

$|L \cap C| = 231$

$|J \cap C| = 290$

$|L \cap J \cap C| = 189$

$|\bar{L} \cap \bar{J} \cap \bar{C}| = ?$

$$|J \cup L \cup C| = |J| + |L| + |C| - |J \cap L| - |L \cap C| - |C \cap J| + |J \cap L \cap C|$$

$$|J \cup L \cup C| = 1876 + 999 + 345 - 876 - 231 - 290 + 189$$

$$= 2{,}012$$

$$|\bar{J} \cap \bar{L} \cap \bar{C}| = |\overline{J \cup L \cup C}| = 2504 - 2012 = 492$$