INDUCTION AND RECURSION

CHAPTER - 5

WEEK - 9

PRINCIPLE OF MATHEMATICAL INDUCTION :-

Let $P(n)$ be a propositional function.

BASIS STEP :- Verify that $P(1)$ is true

INDUCTIVE STEP :- Show that the conditional statement $P(K) \rightarrow P(K+1)$ is true for all positive integers K.

Then $P(n)$ is true for all positive integers n.

NOTE :- 1) To prove the inductive step, assume that $P(K)$ is true for an arbitrary positive integer K and show that under this assumption, $P(K+1)$ must also be true.

2) To prove that $P(K) \rightarrow P(K+1)$ is true for every positive integer K, we need to show that $P(K+1)$ cannot be false when $P(K)$ is true

3) Proof technique can be stated as

$$(P(1) \wedge \forall K\,(P(K) \rightarrow P(K+1))) \rightarrow \forall n\, P(n)$$

where the domain is set of positive integers.

Examples

1) Show that if n is a positive integer then $1+2+\cdots+n = \frac{n(n+1)}{2}$

Solution:- Let $P(n): 1+2+\cdots+n = \frac{n(n+1)}{2}$.

Basis step: Let $n=1$

$LHS = 1$

$RHS = \frac{1(1+1)}{2} = 1$

$LHS = RHS$

$\Rightarrow P(1)$ is true

Induction step: Assume that $P(K)$ is true for arbitrary K.

That is, assume that $1+2+\cdots+K = \frac{K(K+1)}{2}$ — (1)

To show that $P(K+1)$ is true

That is, to show that $1+2+\cdots+(K+1) = \frac{(K+1)(K+1+1)}{2}$

$= \frac{(K+1)(K+2)}{2}$

LHS $1+2+\cdots+(K+1)$

$\Rightarrow \{1+2+\cdots+K\}+(K+1)$

$\Rightarrow \frac{K(K+1)}{2}+K+1$ ($\because$ By (1))

$\Rightarrow \frac{K(K+1)+2(K+1)}{2}$

$\Rightarrow \frac{(K+1)(K+2)}{2}$

RHS

$\frac{(K+1)(K+2)}{2}$.

LHS = RHS

$\Rightarrow$ $P(K+1)$ is true.

$\Rightarrow$ $P(n)$ is true for all $n \in Z^+$

(2) Use Mathematical Induction and show that

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} \text{ for all } n \in Z^+$$

Sol Let $P(n)$: $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

Basis Step:- Let $n = 1$

$$LHS = 1^3 = 1$$

$$RHS = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$$

LHS = RHS

$\Rightarrow$ $P(1)$ is true

Inductive Step:- Assume that $P(K)$ is true for arbitrary K.

That is, assume that $1^3 + 2^3 + \cdots + K^3 = \frac{K^2(K+1)^2}{4}$ — (1)

To show that $P(K+1)$ is true

That is, to show that

$$1^3 + 2^3 + \cdots + (K+1)^3 = \frac{(K+1)^2(K+1+1)^2}{4}$$

$$= \frac{(K+1)^2(K+2)^2}{4}.$$

LHS

$$1^3 + 2^3 + \cdots + (k+1)^3$$

$$\Rightarrow \{1^3 + 2^3 + \cdots + k^3\} + (k+1)^3$$

$$\Rightarrow \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad (\because \text{By } 1)$$

$$\Rightarrow \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$\Rightarrow \frac{(k+1)^2(k^2 + 4(k+1))}{4}$$

$$\Rightarrow \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$\Rightarrow \frac{(k+1)^2(k+2)^2}{4}$$

RHS $\frac{(k+1)^2(k+2)^2}{4}$

LHS = RHS

$\Rightarrow P(k+1)$ is true.

$\Rightarrow P(n)$ is true for all $n \in Z^+$

(3) Prove that $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{ar^n - a}{r-1}$

where $r \neq 1$, $n \in Z^+$

Sol Let $P(n)$: $a + ar + \cdots + ar^{n-1} = \frac{ar^n - a}{r-1}$

Basis Step:- Let $n=1$

$LHS = a$ | RHS $\frac{ar^1 - a}{r-1} = \frac{a\cancel{(r-1)}}{\cancel{(r-1)}} = a$

$LHS = RHS$

$\Rightarrow P(1)$ is true.

Inductive Step:- Assume that $P(K)$ is true for some arbitrary K.

That is, $a + ar + ar^2 + \cdots + ar^{K-1} = \frac{ar^K - a}{r-1}$ — (1)

To show that $P(K+1)$ is true

That is, to show that

$$a + ar + \cdots + ar^{(K+1)-1} = \frac{ar^{K+1} - a}{r-1}$$

LHS

$a + ar + \cdots + ar^{(K+1)-1}$

$\Rightarrow a + ar + \cdots + ar^K$

$\Rightarrow \{a + ar + \cdots + ar^{K-1}\} + ar^K$

$\Rightarrow \frac{ar^K - a}{r-1} + ar^K$ ($\because$ By (1))

$$\Rightarrow \frac{ar^k - a + (r-1)ar^k}{(r-1)}$$

$$\Rightarrow \frac{a\cancel{r^k} - a + ar^{k+1} - a\cancel{r^k}}{r-1}$$

$$\Rightarrow \frac{ar^{k+1} - a}{r-1}$$

$$RHS = \frac{ar^{k+1} - a}{r-1}$$

$LHS = RHS$

$\Rightarrow P(k+1)$ is true

$\Rightarrow P(n)$ is true for all $n \in Z^+$

4) Prove that $1\cdot 2 + 2\cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{Z}^+$

Sol Let $P(n)$: $1\cdot 2 + 2\cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Basis Step:- let $n=1$

$$\text{L.HS} = 1\cdot 2 = 2 \quad \Big| \quad \text{RHS} \; \frac{1(1+1)(1+2)}{3} = \frac{6}{3} = 2$$

LHS = RHS

$\Rightarrow P(1)$ is true.

Inductive Step:- Assume that $P(K)$ is true for some arbitrary K.

That is, $1\cdot 2 + 2\cdot 3 + \cdots + K(K+1) = \frac{K(K+1)(K+2)}{3}$ —— (1)

To prove that $P(K+1)$ is true

That is, to prove that

$$1\cdot 2 + 2\cdot 3 + \cdots + (K+1)(K+1+1) = \frac{(K+1)(K+1+1)(K+1+2)}{3}$$

$$\Rightarrow 1\cdot 2 + 2\cdot 3 + \cdots + (K+1)(K+2) = \frac{(K+1)(K+2)(K+3)}{3}.$$

LHS

$1\cdot 2 + 2\cdot 3 + \cdots + (K+1)(K+2)$

$\Rightarrow \{1\cdot 2 + 2\cdot 3 + \cdots + K(K+1)\} + (K+1)(K+2)$

$\Rightarrow \frac{K(K+1)(K+2)}{3} + (K+1)(K+2)$ ($\because$ By (1))

$$\Rightarrow \frac{K(K+1)(K+2) + 3(K+1)(K+2)}{3}$$

$$\Rightarrow \frac{(K+1)(K+2)(K+3)}{3}$$

$$RHS = \frac{(K+1)(K+2)(K+3)}{3}.$$

$\Rightarrow$ LHS = RHS

$\Rightarrow P(K+1)$ is true

$\Rightarrow P(n)$ is true for all $n \in Z^+$

(5) Prove that $2 + 2\cdot2^2 + 3\cdot2^3 + 4\cdot2^4 + \cdots + n\cdot2^n = (n-1)2^{n+1} + 2$

for all $n \in Z^+$

Sol Let $P(n)$: $2 + 2\cdot2^2 + 3\cdot2^3 + \cdots + n\cdot2^n = (n-1)2^{n+1} + 2$

Basis Step:- Let $n = 1$

LHS $= 2$ | RHS ~~$(1-1)2^{1+1}$~~

$(1-1)2^{1+1} + 2$

$\Rightarrow 0 + 2$

$\Rightarrow 2$

LHS = RHS

$\Rightarrow P(1)$ is true

Inductive Step:-

Assume that $P(K)$ is true for some arbitrary K

That is $2+2\cdot 2^2+3\cdot 2^3+ \cdots + K\cdot 2^K = (K-1)2^{K+1}+2$ — (1)

To prove that $P(K+1)$ is true

That is to prove that

$$2+2\cdot 2^2+3\cdot 2^3+ \cdots +(K+1)2^{K+1} = (K+1-1)2^{K+1+1}+2$$

$$= K2^{K+2}+2$$

LHS

$$2+2\cdot 2^2+3\cdot 2^3+ \cdots +(K+1)2^{K+1}$$

$$\Rightarrow \{2+2\cdot 2^2+3\cdot 2^3+ \cdots + K2^K\} + (K+1)2^{K+1}$$

$$\Rightarrow (K-1)2^{K+1}+2 \quad +(K+1)2^{K+1} \quad (\because \text{By } (1))$$

$$\Rightarrow 2^{K+1}(K-1+K+1)+2$$

$$\Rightarrow 2^{K+1}(2K)+2$$

$$\Rightarrow K2^{K+2}+2$$

RHS $K2^{K+2}+2$

LHS = RHS

$\Rightarrow P(K+1)$ is true

$\Rightarrow P(n)$ is true for all $n \in Z^+$.

(6) Prove that 2 divides n^2+n where $n \in Z^+$

<u>Sol</u> Let $P(n)$: 2 divides n^2+n

<u>Basis step</u> Let $n=1$

$P(1)$ is true because $1^2+1=2$ is divisible by 2.

<u>Inductive step</u>:- Assume that $P(K)$ is true for some arbitrary K.

That is. 2 divides K^2+K.

$$\Rightarrow K^2+K = 2Q \text{ (say)} \quad \text{--- (1)}$$

$$\Rightarrow K^2 = 2Q-K$$

To Prove that $P(K+1)$ is true

That is, to Prove that 2 divides $(K+1)^2+(K+1)$

<u>Consider</u> $(K+1)^2+(K+1) = K^2+3K+2$

$$= (2Q-K)+3K+2$$

$$= 2Q+2K+2$$

$$= 2(Q+K+1)$$

Clearly $(K+1)^2+(K+1)$ is divisible by 2

$\Rightarrow P(K+1)$ is true

$\Rightarrow P(n)$ is true for all $n \in Z^+$

7. Prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for $n \in Z^+$

Sol Let $P(n): 7^{n+2} + 8^{2n+1}$ is divisible by 57

Basis step:- Let $n = 1$.

P(1) is true because $7^{1+2} + 8^{2+1}$

$\Rightarrow 7^3 + 8^3$

$= 343 + 512$

$= 855$ is divisible by 57

Inductive step:- Assume that P(K) is true for some arbitrary k.

That is $7^{K+2} + 8^{2K+1}$ is divisible by 57

$\Rightarrow 7^{K+2} + 8^{2K+1} = 57q$ (say)

$\Rightarrow 7^{K+2} = 57q - 8^{2K+1}$ —— (1)

To prove that P(K+1) is true.

That is, to prove that $7^{(K+1)+2} + 8^{2(K+1)+1}$ is divisible by 57

Consider $7^{(K+1)+2} + 8^{2(K+1)+1}$

$= 7^{K+3} + 8^{2K+3}$

$= 7^{K+2}\,7 + 8^{2K+3}$

$= (57q - 8^{2K+1})7 + 8^{2K+1}8^2$ (∵ by (1))

$= 7\cdot 57q - 7\cdot 8^{2K+1} + 64\cdot 8^{2K+1}$

$= 7\cdot 57q + 57\cdot 8^{2K+1}$

$= 57(7q + 8^{2K+1})$

clearly $7^{k+1+2} + 8^{2(k+1)+1}$ is divisible by 57

$\Rightarrow P(k+1)$ is true

$\Rightarrow P(n)$ is true for all $n \in Z^+$

8) Prove that 5 divides $n^5 - n$ where $n \in Z^+$

Sol: Let $P(n)$: 5 divides $n^5 - n$

Basis Step:- Let $n=1$
$P(1)$ is true because $1^5 - 1$
$= 1-1$
$= 0$ is divisible by 5

Inductive Step:- Assume that $P(k)$ is true for some arbitrary k

That is, 5 divides $k^5 - k$

$\Rightarrow k^5 - k = 5q$ (say)

$\Rightarrow k^5 = 5q + k$ —— (1)

To prove that $P(k+1)$ is true

That is, 5 divides $(k+1)^5 - (k+1)$

Consider

$(k+1)^5 - (k+1)$

$\Rightarrow (k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) - (k+1)$

$\Rightarrow k^5 + 5k^4 + 10k^3 + 10k^2 + 4k$

$\Rightarrow (5q + k) + 5k^4 + 10k^3 + 10k^2 + 4k$ ($\because$ By (1))

$\Rightarrow 5q + 5k^4 + 10k^3 + 10k^2 + 5k$

$\Rightarrow 5(q + k^4 + 2k^3 + 2k^2 + 1)$

Clearly this is divisible by 5

$\Rightarrow$ $P(K+1)$ is true

$\Rightarrow$ $P(n)$ is true for all $n \in Z^+$

STRONG INDUCTION:- Let $P(n)$ be a propositional function.

Basis Step:- Verify $P(1)$ is true.

Inductive Step:- Show that the conditional statement

$$(P(1) \wedge P(2) \wedge \ldots \wedge P(K)) \rightarrow P(K+1) \text{ is true.}$$

for all positive integers K

Then $P(n)$ is true for all positive integers n.

THE WELL-ORDERING PROPERTY :-

Every nonempty set of nonnegative integers has a least element.

RECURSION:- Sometimes it is difficult to define an object explicitly. However, it may be easy to define this object in terms of itself. This process is called Recursion.

Recursion can be used to define sequences, functions and sets.

For example $a_n = 2^n$ for $n = 0, 1, 2, \ldots$

This can also be defined by giving the first term of sequence namely $a_0 = 1$ and a Rule for finding a term of sequence from previous one, namely $a_{n+1} = 2a_n$ for $n = 0, 1, 2, 3, \ldots$

RECURSIVE (OR) INDUCTIVE DEFINITION :- (Functions)

Basis step :- Specify the value of the function at zero.

Inductive step :- Give a rule for finding the value of the function at an integer from its values at smaller integers.

Examples: 1) Give a recursive definition of a^n where $a \neq 0 \in \mathbb{R}$ and $n \in \mathbb{Z}^+$

Sol. Basis step : $a^0 = 1$

Recursive (or) Inductive step : $a^{n+1} = a\,a^n$ for $n = 0, 1, 2, \ldots$

· Recursive definition (Sets)

Basis step :- Specify an initial collection of elements.

Inductive step :- Give a rule for forming new elements in the set from those already known to be in the set.

Example : Consider the subset S of the set of integers recursively defined by

Basis step : $3 \in S$

Inductive step : If $x \in S$ and $y \in S$, then $x + y \in S$.

$S = \{3, 3+3=6, 3+6=9, 6+6=12, 3+9=12, 6+9=15, \ldots\}$

$S =$ Set of all positive multiples of 3

Examples

1. find $f(1), f(2), f(3)$ and $f(4)$ if $f(n)$ is defined recursively by $f(0)=1$ and

a) $f(n+1) = f(n)+2$ b) $f(n+1) = 3f(n)$

c) $f(n+1) = 2^{f(n)}$ d) $f(n+1) = f(n)^2 + f(n)+1$

Sol a) Given $f(0)=1$

$f(n+1) = f(n)+2$

Put $n=0, 1, 2, 3$

$f(1) = f(0)+2 = 1+2 = 3$

$f(2) = f(1)+2 = 3+2 = 5$

$f(3) = f(2)+2 = 5+2 = 7$

$f(4) = f(3)+2 = 7+2 = 9$

b) Given $f(0)=1$

$f(n+1) = 3f(n)$

Put $n=0, 1, 2, 3$

$f(1) = 3f(0) = 3 \cdot 1 = 3$

$f(2) = 3f(1) = 3 \cdot 3 = 9$

$f(3) = 3f(2) = 3 \cdot 9 = 27$

$f(4) = 3f(3) = 3 \cdot 27 = 81$

c) Given $f(0)=1$

$f(n+1) = 2^{f(n)}$

Put $n=0, 1, 2, 3$

$f(1) = 2^{f(0)} = 2^1 = 2$

$f(2) = 2^{f(1)} = 2^2 = 4$

$f(3) = 2^{f(2)} = 2^4 = 16$

$f(4) = 2^{f(3)} = 2^{16} = 65536$

d) Given $f(0)=1$

$f(n+1) = f(n)^2 + f(n)+1$

Put $n=0, 1, 2, 3$

$f(1) = f(0)^2 + f(0)+1$

$= 1^2+1+1 = 3.$

$f(2) = f(1)^2 + f(1)+1$

$= 3^2+3+1 = 13$

$f(3) = f(2)^2 + f(2)+1$

$= 13^2+13+1 = 183$

$f(4) = f(3)^2 + f(3)+1$

$= 183^2 + 183 + 1$

$= 33673$

(2) Give a recursive definition of the sequence $\{a_n\}$ where $n = 1, 2, 3, \ldots$ if

a) $a_n = 6n$ b) $a_n = 2n+1$ c) $a_n = 10^n$ d) $a_n = 5$

Sol a) $a_1 = 6$
$a_{n+1} = 6(n+1) = 6n+6$
$\Rightarrow a_{n+1} = a_n + 6$

b) $a_1 = 3$
$a_{n+1} = 2(n+1)+1$
$= (2n+1)+2$
$= a_n + 2$

c) $a_1 = 10$
$a_{n+1} = 10^{n+1}$
$= 10\, a_n$

d) $a_1 = 5$
$a_{n+1} = 5$
$= a_n$

(3) Give a recursive definition of the set of positive integers that are multiples of 5

Sol. Basis step: $5 \in S$

Recursive Step:- If $x \in S, y \in S$ then $x+y \in S$.

$S = \{5, 5+5=10, 5+10=15, 10+10=20, \ldots\}$
$= \{5, 10, 15, \ldots\}$.

(4) Give a recursive definition of the set of even integers.

Sol Basis step: $0 \in S$

Inductive step: If $x \in S$, then $x+2 \in S$ and $x-2 \in S$

$S = \{0, 0+2=2, 0-2=-2, 2+2=4, (-2)+(-2)=-4, \ldots\}$
$= \{0, 2, -2, 4, -4, \ldots\}$.

(5) Odd positive integers: Basis step: $1 \in S$

Inductive step: If $x \in S$, then $2x+1 \in S$ and $2x-1 \in S$