Chapter - 1 (Summary)

The Foundations: Logic and Proofs - I

Week - 2

Proposition:- A sentence that is either true or false but not both

Eg Toronto is the capital of Canada. (False)

$1+1=2$ (True)

What time is it? (Not a Proposition)

$x+1=2$ (Not a Proposition)

Truth value of a Proposition

- True (T)
- False (F)

Logical operators (Connectives)

Negation:- The negation of Proposition p is denoted by $\neg p$ is the statement "It is not the case that p"

Truth Table

p	$\neg p$
T	F
F	T

Eg 1) p = Micheal P.C runs Linux

$\neg p$ = Micheal P.C does not run Linux.

Eg 2) p = Salman Smart Phone has atleast 16 GB of memory

$\neg p$ = Salman Smart Phone has less than 16 GB of memory

Conjunction:- The conjunction of two propositions p and q is denoted by "$p \wedge q$" is the proposition "p and q"

Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The conjunction $p \wedge q$ is true when both p and q are true and false otherwise.

Disjunction:- The disjunction of two propositions p and q is denoted by "$p \vee q$" is the proposition "p or q"

Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The disjunction $p \vee q$ is false when both p and q are false and true otherwise.

Exclusive or:- The exclusive or of two propositions p and q is denoted by "$p \oplus q$" is the proposition which is true when exactly one of p and q is true and false otherwise

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

NOTE:- To remember this understand the meaning of soup or salad comes with the order

(Implication)

Conditional Statement: The conditional statement of two proposition p and q is denoted by $p \to q$, is the proposition "if p then q"

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$p \to q$ is false if p is true and q is false, and true otherwise.

NOTE: understanding Implication

If I am elected then I will lower the taxes

Bi conditional Statement:- The Bi conditional statement of two proposition p and q is denoted by $p \leftrightarrow q$, is the proposition "p if and only if q"

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$p \leftrightarrow q$ is true if both p and q have same truth values, and false otherwise.

Converse, Contrapositive and Inverse.

If $p \to q$ is a conditional statement

then Converse is $q \to p$

Contrapositive is $\neg q \to \neg p$

Inverse is. $\neg p \to \neg q$

Logical Equivalent: When two compound propositions have same truth value, then they are said to be logically equivalent

Example 1

Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent (i.e, Conditional and Contrapositive).

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The truth values of $p \to q$ and $\neg q \to \neg p$ are same in all cases, Hence they are logically equivalent

Example 2 Show that $q \to p$ and $\neg p \to \neg q$ are logically equivalent (i.e, Converse and Inverse)

q	p	$q \to p$	$\neg p$	$\neg q$	$\neg p \to \neg q$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The truth values of $q \to p$ and $\neg p \to \neg q$ are same in all cases. Hence they are logically equivalent.

TAUTOLOGY:- A compound proposition which is always true whatever might be the truth values of the propositional variables.

Eg: $p \vee \neg p$.

CONTRADICTION:- Always false.

Eg: $p \wedge \neg p$

CONTINGENCY:- Neither Tautology nor contradiction.

Example:- Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

Sol

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Example. Show that $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$ is a tautology.

Sol

p	q	r	$\neg p$	$p \vee q$	$(\neg p \vee r)$	$(p \vee q) \wedge (\neg p \vee r)$	$(q \vee r)$	$(p \vee q) \wedge (\neg p \vee r) \rightarrow q \vee r$
T	T	T	F	T	T	T	T	T
T	T	F	F	T	F	F	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	F	T	T	F	T	F	T	T
F	F	F	T	F	T	F	F	T

Examples 1) Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

2) Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

3) Show that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent

4) Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent

5) Show that $p \to q$ and $\neg p \vee q$ are logically equivalent

6) Show that $p \vee (q \wedge r)$ and $(p \wedge q) \wedge (p \vee r)$ are logically equivalent.

Logic and Bit operations

Bits → 1 (T)
Bits → 0 (F)

Bit operators

OR ($\vee$)
AND ($\wedge$)
XOR ($\oplus$)

x	y	x OR y	x AND y	x XOR y
1	1	1	1	0
1	0	1	0	1
0	1	1	0	1
0	0	0	0	0

Bit string : A bit string is a sequence of zero or more bits

Bit operations to Bit strings

0 1 1 0 1 1 0 1 1 0
1 1 0 0 0 1 1 1 0 1

Bitwise OR → 1 1 1 0 1 1 1 1 1 1
Bitwise AND → 0 1 0 0 0 1 0 1 0 0
Bitwise XOR → 1 0 1 0 1 0 1 0 1 1

DEMORGAN LAWS

1) $\neg(p \vee q) = \neg p \wedge \neg q$

That means negation of disjunction of two Propositions is Conjunction of negation of Individual propositions.

2) $\neg(p \wedge q) = \neg p \vee \neg q$

Means that Negation of Conjunction of two Propositions is disjunction of negation of Individual Propositions.

Other important Properties:

1) $\neg(\neg p) = p$

2) Negation Laws:

(i) $p \vee \neg p \equiv T$

(ii) $p \wedge \neg p \equiv F$

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

3) Absorption Laws.

(i) $p \vee (p \wedge q) \equiv p$

(ii) $p \wedge (p \vee q) \equiv p$

p	q	$p \wedge q$	$p \vee q$	$p \vee (p \wedge q)$	$p \wedge (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	T	F	F
F	F	F	F	F	F

These truth values coincide with p.

Example. Translate the following Sentence to Propositional logic.

"If I go to Dammam or Jeddah, I will not go to Shopping"

Sol:- If I go to Dammam or Jeddah, I will not go to Shopping

If p or q then not r where p = I go to Dammam

q = I go to Jeddah

r = I will go to Shopping.

$\Rightarrow (p \vee q) \rightarrow \neg r$

Model Questions

True/False Type

1) $y \vee \neg(\neg x \wedge y)$ is a Tautology — True

2) The compound propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are not equivalent. — False

Reason
De morgan law $\neg(p \wedge q) = \neg p \vee \neg q$ (Equivalent)

3) The negation of conjunction of two propositions is equivalent to the disjunction of the negation of those propositions. — True

$$\left\{\begin{array}{l}\neg(p \wedge q) = \neg p \vee \neg q \\ \neg(p \vee q) = \neg p \wedge \neg q\end{array}\right.$$

Reason

4) The conditional statement $p \to q$ is false when both p and q are false — False

5) $p \vee \neg p = F$ — False

6) $p \wedge \neg p = F$ — True

MCQ

1) The compound proposition $p \wedge q$ is equivalent to

A) $p \to q$ ✓B) $\neg(p \to \neg q)$ C) $\neg p \to q$ D) None

2) The converse statement of the compound proposition $q \to p$ is

A) $\neg p \to \neg q$ B) $\neg q \to \neg p$ ✓C) $p \to q$ D) None

3) The compound proposition $p \vee (p \wedge q)$ is equivalent to

✓A) p B) q C) $p \vee q$ D) $p \wedge q$

4) The converse of the proposition "If home team wins then it is raining" is

A) If it is not raining, then the home team does not win
B) If the home team wins, then it is not raining
C) If the home team does not win, then it is not raining
✓D) If it is raining then home team wins.

Essay Type

1) Construct the truth table of compound proposition $(p \to q) \wedge (\neg p \to r)$

p	q	r	$\neg p$	$p \to q$	$\neg p \to r$	$(p \to q) \wedge (\neg p \to r)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	F	T	F
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	F

2) Find the contrapositive and inverse statements of the statement "If you do every exercise in this book then you are a good student"

Sol. If you do every exercise in this book then you are a good student

$\Rightarrow$ If p then q.

p = You do every exercise in this book

q = You are a good student.

Contrapositive of $p \to q$

$$\neg q \to \neg p$$

That means if $\neg q$ then $\neg p$

$\Rightarrow$ If you are not a good student then you don't do every exercise in this book.

Inverse of $p \to q$

$$\neg p \to \neg q$$

That means if $\neg p$ then $\neg q$

$\Rightarrow$ If you don't do every exercise in this book then you are not a good student.